Pure Spinor Formulation of the Superstring and Its Applications

超弦的纯旋量表述及其应用

Nathan Berkovits and Carlos R. Mafra

内森·贝科维茨与卡洛斯·R·马夫拉

Contents

目录

Abstract

摘要

The pure spinor formalism for the superstring has the advantage over the more conventional Ramond-Neveu-Schwarz formalism of being manifestly spacetime supersymmetric, which simplifies the computation of multiparticle and multiloop amplitudes and allows the description of Ramond-Ramond backgrounds. In addition to the worldsheet variables of the Green-Schwarz-Siegel action, the pure spinor formalism includes bosonic ghost variables which are constrained spacetime spinors and are needed for covariant quantization using a nilpotent BRST operator.

相较于更传统的 Ramond-Neveu-Schwarz 形式，超弦的纯旋子形式具有时空超对称明显化的优势，这简化了多粒子与多圈振幅的计算，还能够描述 Ramond-Ramond 背景。除 Green-Schwarz-Siegel 作用量的世界面变量外，纯旋子形式还引入了玻色鬼变量，这类变量是受约束的时空旋子，使用幂零 BRST 算符进行协变量子化时需要它们。

N. Berkovits (☑)

N. Berkovits (☑)

ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, UNESP - Universidade Estadual Paulista, São Paulo, SP, Brazil

巴西圣保罗州 UNESP 圣保罗州立大学理论物理研究所 ICTP 南美基础研究所

e-mail: nathan.berkovits@unesp.br

电子邮箱:nathan.berkovits@unesp.br

C. R. Mafra

C. R. Mafra

STAG Research Centre and Mathematical Sciences, University of Southampton, Southampton, UK

英国南安普顿南安普顿大学数学与 STAG 研究中心

e-mail: c.r.mafra@soton.ac.uk

电子邮箱:c.r.mafra@soton.ac.uk

In this review, several applications of the formalism are described including the explicit computation in D = 10 superspace of the general disk amplitude with an arbitrary number of external massless states, genus one amplitudes with up to seven external states, genus-two amplitudes with up to five external states, and the

low-energy limit of the genus-three amplitude with up to four external states. The pure spinor formalism has also been used to covariantly quantize the superstring in an $AdS_5 \times S^5$ background and might be useful for proving the AdS-CFT correspondence in the limit of small AdS radius.

本综述介绍了该形式的若干应用，包括在 D = 10 超空间中明确计算任意多外部无质量态的一般圆盘振幅、至多七个外部态的亏格 1 振幅、至多五个外部态的亏格 2 振幅，以及至多四个外部态亏格 3 振幅的低能极限。纯旋子形式还被用于对 $AdS_5 \times S^5$ 背景下的超弦进行协变量子化，或对证明 AdS 半径很极限下的 AdS-CFT 对应有所帮助。

This is an overview written for the "Handbook of Quantum Gravity," eds. C. Bambi, L. Modesto and I. Shapiro.

本文是为《量子引力手册》撰写的综述，手册主编为 C. Bambi、L. Modesto 和 I. Shapiro。

Keywords

关键词

Superstring - Supersymmetry - Amplitudes - Pure spinors

超弦 - 超对称 - 振幅 - 纯旋子

Introduction

引言

At the present time, superstring theory is the only formalism available for computing perturbative scattering amplitudes of gravitons without ultraviolet quantummechanical divergences. Although comparing these scattering amplitudes with experiments is unlikely in the near future, various properties of these amplitudes such as spacetime supersymmetry and duality symmetry might have testable low-energy implications.

目前，超弦理论是唯一可用于计算引力子微扰散射振幅、且不存在紫外量子力学发散的形式体系。虽然在近期内难以将这些散射振幅与实验对比，但这些振幅的各类性质——例如时空超对称与对偶对称性——或许在低能区存在可检验的推论。

Using the conventional Ramond-Neveu-Schwarz (RNS) formalism of the super-string, the complicated nature of vertex operators for spacetime fermions and the need to sum over spin structures have made it difficult to compute amplitudes involving external fermions or to compute multiloop amplitudes. Furthermore, backgrounds involving Ramond-Ramond fields necessary for the AdS-CFT correspondence are difficult to describe in the RNS formalism.

使用传统的 Ramond-Neveu-Schwarz(RNS) 超弦形式体系时，时空费米子顶点算符性质复杂，且还需要对自旋结构求和，这导致很难计算包含外费米子的振幅，也很难计算多圈振幅。此外，AdS-CFT 对应所需的、包含 Ramond-Ramond 场的背景很难在 RNS 形式体系下描述。

In 2000, a new formalism for the superstring was constructed in which spacetime supersymmetry is manifest and there is no need to sum over spin structures [1]. In addition to the worldsheet variables (x^m, θ^α) of the Green-Schwarz formalism [2] for $m = 0$ to 9 and $\alpha = 1$ to 16, this new formalism includes the fermionic momenta variables d_α of Siegel [3] as well as bosonic ghost variables $(\lambda^\alpha, w_\alpha)$ constrained to satisfy $\lambda\gamma^m\lambda = 0$. This constraint implies that λ^α is a $D = 10$ "pure spinor" as defined by Cartan with 11 independent components, and the conformal anomaly contribution of +22 from $(\lambda^\alpha, w_\alpha)$ cancels the conformal anomaly contribution of $+10 - 32 = -22$ from x^m and $(\theta^\alpha, d_\alpha)$. Generalizing a supersymmetric field-theory observation of Howe [4, 5], physical superstring states in this "pure spinor formalism" are defined using the nilpotent BRST operator $Q = \oint dz \lambda^\alpha d_\alpha$ and, unlike in the Green-Schwarz formalism, covariant quantization is straightforward (Other early applications of D = 10 pure spinors include [88] in off-shell super-Yang-Mills and [89] in classical superstrings.). A similar BRST operator is useful for describing $d = 11$ supergravity [6-8], and more details on pure spinor applications in supersymmetric field theories can be found in the review of Martin Cederwall [9].

2000 年，人们构造出了一种新的超弦形式体系，该体系中时空超对称是明显的，且无需对自旋结构求和 [1]。除了格林-施瓦茨形式体系 [2] 中对应 $m = 0$ 取 1 到 9、$\alpha = 1$ 取 1 到 16 的世界面变量 (x^m, θ^α) 之外，这个新形式体系还包含 Siegel[3] 提出的费米动量变量 d_α ， 以及满足约束条件 $\lambda\gamma^m\lambda = 0$ 的玻色鬼变量 $(\lambda^\alpha, w_\alpha)$ 。该约束表明 λ^α 是 Cartan 定义的 $D = 10$ "纯旋量"，具有 11 个独立分量，其中 $(\lambda^\alpha, w_\alpha)$ 贡献的 +22 共形反常抵消了 x^m 和 $(\theta^\alpha, d_\alpha)$ 贡献的 $+10 - 32 = -22$ 共形反常。推广 Howe[4,5] 在超对称场论中的发现，这种"纯旋量形式体系"的物理超弦态由幂零 BRST 算符 $Q = \oint dz \lambda^\alpha d_\alpha$ 定义，且与格林-施瓦茨形式体系不同，该体系的协变量化十分简便 (D = 10 纯旋量的其他早期应用包括脱壳超杨-米尔斯 [88] 和经典超弦 [89])。一个类似的 BRST 算符可用于描述 $d = 11$ 超引力 [6-8]，关于纯旋量在超对称场论中的应用的更多细节可以参考 Martin Cederwall 的综述 [9]。

Over the last 20 years, this pure spinor formalism has been used to compute various multiparticle and multiloop amplitudes in superstring theory including several amplitudes which have not yet been computed using the RNS formalism. All amplitudes computed using both the RNS and pure spinor formalisms have been shown to coincide; however, the pure spinor computations are typically much more efficient since there is no sum over spin structures and amplitudes are automatically expressed in $D = 10$ superspace. Nevertheless, a proof of equivalence of the RNS and pure spinor formalisms for the superstring is still lacking.

过去 20 年间，纯旋量形式体系已被用于计算超弦理论中各类多粒子、多圈振幅，其中多个振幅至今还无法用 RNS 形式体系完成计算。已经证明，同时用 RNS 和纯旋量形式体系计算的所有振幅结果一致；不过纯旋量计算通常效率高得多，因为不需要对自旋结构求和，且振幅自动表示在 $D = 10$ 超空间中。尽管如此，RNS 形式体系与超弦纯旋量形式体系的等价性证明仍未完成。

A promising approach toward proving this equivalence involves a recently constructed formalism for the superstring which includes θ^α and an unconstrained bosonic spacetime spinor worldsheet variable Λ^α , in addition to the usual N = 1 worldsheet supersymmetric RNS matter and ghost variables [10,11]. This new formalism, named the B-RNS-GSS formalism since it combines features of the RNS, Green-Schwarz-Siegel, and pure spinor formalisms, is both N = 1 worldsheet supersymmetric and D = 10 spacetime supersymmetric and acts as a bridge between the RNS and pure spinor formalisms. It can be related to the RNS formalism by treating $(\theta^\alpha, \Lambda^\alpha)$ as non-minimal variables which decouple from the BRST cohomology, and can be related to the pure spinor formalism by "twisting" the N = 1 superconformal generators into N = 2 superconformal generators so that all worldsheet variables carry integer conformal weight. Work is in progress on computing scattering amplitudes using the B-RNS-GSS formalism and proving that the amplitudes coincide with those computed using the RNS and pure spinor formalisms. Since multiloop amplitude computations using the RNS and pure spinor formalism have different types of subtleties, it is expected that the B-RNS-GSS formalism will be useful for relating these subtleties.

> 证明该等价性的一种可行方法，用到了近期构建的超弦形式体系，它除了常规的 θ^α 世界面超对称 RNS 物质与鬼变量 [10,11] 外，还包含 θ^α 和一个无约束玻色时空旋量世界面变量 Λ^α 。这种新形式体系名为 B-RNS-GSS 形式体系，结合了 RNS、格林-施瓦茨-西格尔和纯旋量形式体系的特点，同时具备 N = 1 世界面超对称性与 D = 10 时空超对称性，可作为 RNS 形式体系和纯旋量形式体系之间的桥梁。它可通过将 $(\theta^\alpha, \Lambda^\alpha)$ 视为从 BRST 上同调中退耦的非最小变量关联到 RNS 形式体系，也可通过将 N = 1 超共形生成元"扭转"为 N = 2 超共形生成元，使所有世界面变量都具有整数共形权重，从而关联到纯旋量形式体系。目前已有研究正在利用 B-RNS-GSS 形式体系计算散射振幅，并证明所得振幅与 RNS 和纯旋量形式体系的计算结果一致。由于 RNS 和纯旋量形式体系的多圈振幅计算各有不同类型的难点，预计 B-RNS-GSS 形式体系将有助于梳理这些难点之间的关联。

Just as the RNS formalism for the superstring can be described in any curved background which preserves N = 1 worldsheet supersymmetry, the pure spinor formalism can be described in any curved background in which the BRST current $\lambda^\alpha d_\alpha$ remains nilpotent and holomorphic [12]. This allows not only the Calabi-Yau backgrounds which can be described using the RNS formalism but also any curved supergravity background in which the D = 10 supergravity equations of motion are satisfied to lowest order in α' . For example, unlike the RNS formalism, the pure spinor formalism can be used to covariantly quantize the superstring in an $AdS_5 \times S^5$ Ramond-Ramond background which is dual to $\mathcal{N} = 4D = 4$ super-Yang-Mills through the AdS-CFT correspondence.

> 正如超弦的 RNS 形式体系可在任何保持 N = 1 世界面超对称性的弯曲背景下描述，纯旋量形式体系可在任何满足 BRST 流 $\lambda^\alpha d_\alpha$ 保持幂零性与全纯性的弯曲背景下描述 [12]。该形式体系不仅能描述 RNS 形式体系可处理的卡拉比-丘背景, 还能处理所有满足 D = 10 超引力运动方程在 α' 一阶近似下成立的弯曲超引力背景。例如, 与 RNS 形式体系不同, 纯旋量形式体系可用于对 $AdS_5 \times S^5$ 拉姆齐-拉姆齐 (Ramond-Ramond) 背景下的超弦进行协变量子化，该背景通过 AdS-CFT 对偶与 $\mathcal{N} = 4D = 4$ 超杨-米尔斯理论对偶。

Although this important application will not be discussed in later sections of the review, quantum consistency of the $AdS_5 \times S^5$ background has been proven [13] using the pure spinor formalism. To prove quantum consistency to all orders in α' , it was shown using symmetry arguments that any potential BRST anomalies coming from quantum corrections can be cancelled by the addition of local counterterms to the worldsheet action. It was also shown using BRST arguments that the classical nonlocal conserved currents related to integrability can be extended to quantum nonlocal conserved currents.

> 尽管这一重要应用不会在本综述的后续章节讨论, 但已有研究利用纯旋量形式体系证明了 $AdS_5 \times S^5$ 背景的量子一致性 [13]。为了证明该一致性在 α' 的所有阶下都成立，研究通过对称性论证表明: 量子修正带来的所有潜在 BRST 反常都可以通过给世界面作用量添加局部抵消项来消除。同时也通过 BRST 论证表明: 经典下与可积性相关的非局域守恒流可以推广为量子非局域守恒流。

The construction of BRST-invariant vertex operators for half-BPS states in an $AdS_5 \times S^5$ background was recently achieved [14-16], and work is in progress on using these vertex operators for the computation of scattering amplitudes. The structure of the vertex operators and the pure spinor worldsheet action in an $AdS_5 \times S^5$ background are more complicated than in a flat background; however, the manifest $PSU(2,2 \mid 4)$ isometry of the construction should be useful in simplifying the amplitude computations. An important open question is how to generalize the super-Poincaré-invariant BRST cohomology methods which are described in this review to BRST cohomology methods with $PSU(2,2 \mid 4)$ invariance.

近期研究已成功构造了 $AdS_5 \times S^5$ 背景下半 BPS 态的 BRST 不变顶点算子 [14-16]，目前正利用这些顶点算子开展散射振幅计算工作。$AdS_5 \times S^5$ 背景下，顶点算子与纯旋量世界面作用量的结构比平坦背景下更复杂；但该构造的明显 $PSU(2,2 \mid 4)$ 等距性应当有助于简化振幅计算。一个悬而未决的重要问题是: 如何将本文综述介绍的超庞加莱不变 BRST 上同调方法推广为满足 $PSU(2,2 \mid 4)$ 不变性的 BRST 上同调方法。

In the limit of small AdS radius, the pure spinor version of the $AdS_5 \times S^5$ worldsheet action has been shown to reduce to a BRST-trivial topological action plus a small $PSU(2,2 \mid 4)$ -invariant deformation term [17,18]. In this limit, the dual theory is $\mathcal{N} = 4D = 4$ super-Yang-Mills at weak coupling, and it has been conjectured that the topological action describes free super-Yang-Mills and the deformation describes the cubic super-Yang-Mills interaction term. The topological action and deformation term are constructed by combining the x^m and λ^α bosonic worldsheet variables of the pure spinor formalism into a twistor-like variable which transforms linearly under the $SO(4,2) \times SO(6)$ bosonic subgroup of $PSU(2,2 \mid 4)$. Similar twistor variables have been extremely useful for computing perturbative scattering amplitudes of $\mathcal{N} = 4D = 4$ super-Yang-Mills [19], and it would not be surprising if the two types of twistor variables are related through the AdS-CFT correspondence.

在 AdS 半径很小的极限下，已证明 $AdS_5 \times S^5$ 世界面作用量的纯旋量版本可约化为一个 BRST 平凡拓扑作用量加上一个小的 $PSU(2,2 \mid 4)$ 不变形变项 [17,18]。在此极限下，对偶理论是弱耦合的 $\mathcal{N} = 4D = 4$ 超杨-米尔斯理论，且有猜想认为该拓扑作用量描述自由超杨-米尔斯，形变项描述三次超杨-米尔斯相互作用项。拓扑作用量和形变项的构造方法是将纯旋量形式体系的 x^m 和 λ^α 玻色世界面变量组合为一个类扭量变量，该变量在 $PSU(2,2 \mid 4)$ 的 $SO(4,2) \times SO(6)$ 玻性子群下线性变换。类似的类扭量变量对计算 $\mathcal{N} = 4D = 4$ 超杨-米尔斯的微扰散射振幅极为有用 [19]，且这两类扭量变量通过 AdS-CFT 对应联系起来并不令人意外。

If this conjecture could be verified, it would provide a proof of the AdS-CFT correspondence in the case of $AdS_5 \times S^5$. A proof of the AdS-CFT correspondence in the simpler case of $AdS_3 \times S^3$ was established by Eberhardt, Gaberdiel, and Gopakumar in [20] using a ”hybrid” formalism of the superstring which can be interpreted as a six-dimensional version of the $D = 10$ pure spinor formalism. It is very suggestive that twistor-like variables were used in their proof and that Gaberdiel and Gopakumar were recently able to generalize their twistor-like construction of the spectrum of $AdS_3 \times S^3$ at zero radius to the more interesting case of $AdS_5 \times S^5$ at zero radius [21].

如果这一猜想得到验证，就能为 $AdS_5 \times S^5$ 情形下的 AdS-CFT 对应提供证明。Eberhardt、Gaberdiel 和 Gopakumar 在文献 [20] 中利用超弦的“混合”形式体系证明了更简单的 $AdS_3 \times S^3$ 情形下的 AdS-CFT 对应，该混合形式可解释为 $D = 10$ 纯旋量形式体系的六维版本。他们的证明中使用了类扭量变量，且 Gaberdiel 和 Gopakumar 最近成功将零半径下 $AdS_3 \times S^3$ 谱的类扭量构造推广到更受关注的零半径下 $AdS_5 \times S^5$ 情形 [21]，这一结果很有启发性。

After a brief review of the pure spinor formalism and the superspace formulation of ten-dimensional super-Yang-Mills theory in sections ”Ten-Dimensional Super-Yang-Mills Theory in Superspace”, and ”Non-minimal Pure Spinor Formalism”, section ”Superstring Amplitudes with Pure Spinors” will showcase its applications to the computation of scattering amplitudes in a flat background. From the complete genus-zero amplitudes with an arbitrary number of external massless states to the low-energy limit of the massless four-point amplitude at genus three, the pure spinor formalism and related techniques played a crucial role in determining their manifestly supersymmetric forms. Finally, section ”Verifying S-Duality Conjectures” will discuss how these amplitudes have been used to test S-duality conjectures.

在“超空间中的十维超杨-米尔斯理论”和“非最小纯旋量形式体系”两节简要回顾纯旋量形式体系与十维超杨-米尔斯理论的超空间表述后，“纯旋量处理超弦振幅”一节将展示该形式体系在平坦背景下散射振幅计算中的应用。从任意数外 massless 态的完整亏格零振幅，到亏格三时四点 massless 振幅的低能极限，纯旋量形式体系及相关方法在确定它们的显式超对称形式中发挥了关键作用。最后，“验证 S 对偶猜想”一节将讨论如何利用这些振幅检验 S 对偶猜想。

The Pure Spinor Formalism and Scattering Amplitudes

纯旋量形式论与散射振幅

Ten-Dimensional Super-Yang-Mills Theory in Superspace

超空间中的十维超杨-米尔斯理论

There is a super-Poincaré description of $D = 10$ super-Yang-Mills (SYM) in superspace [79,80] that describes the gluon and gluino states via Lie algebra-valued superfield connections $\mathbb{A}_\alpha(x,\theta)$ and $\mathbb{A}_m(x,\theta)$ satisfying the nonlinear constraint $\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m$, where $\nabla_\alpha = D_\alpha - \mathbb{A}_\alpha$ and $\nabla_m = \partial_m - \mathbb{A}_m$ are supercovariant derivatives and

超空间中存在 $D = 10$ 超杨-米尔斯 (SYM) 的超庞加莱描述 [79,80]，该描述通过李代数取值的超场联络 $\mathbb{A}_\alpha(x,\theta)$ 和 $\mathbb{A}_m(x,\theta)$ 描述胶子与胶戈西诺态，二者满足非线性约束 $\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m$ ，其中 $\nabla_\alpha = D_\alpha - \mathbb{A}_\alpha$ 和 $\nabla_m = \partial_m - \mathbb{A}_m$ 是超协变导数,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m \tag{1}$$

is the superspace derivative satisfying $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$. The ten-dimensional superspace coordinates (x,θ) are composed of a $SO(9,1)$ Lorentz vector x^m , where $m = 1,\dots,10$, and a Weyl spinor θ^α , where $\alpha = 1,\dots,16$. In ten dimensions, the Lorentz group has two inequivalent spinor representations, denoted Weyl and anti-Weyl. They are distinguished by the position of the spinor index, upstairs for Weyl Ψ^α and downstairs for anti-Weyl χ_α which cannot be raised or lowered. The gamma matrices $\gamma^m_{\alpha\beta}$ and $\gamma_m^{\alpha\beta}$ are the 16×16 off-diagonal symmetric Pauli matrices of the 32×32 Dirac matrices Γ^m of the $SO(9,1)$ Clifford algebra $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{1}_{32\times32}$. They satisfy $\gamma^m_{\alpha\beta}(\gamma^n)^{\beta\rho} + \gamma^n_{\alpha\beta}(\gamma^m)^{\beta\rho} = 2\eta^{mn}\delta^\rho_\alpha$.

是满足 $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$ 的超空间导数。十维超空间坐标 (x,θ) 由一个 $SO(9,1)$ 洛伦兹向量 x^m 组成，其中 $m = 1,\ldots,10$ ，以及一个外尔旋量 θ^α ，其中 $\alpha = 1,\ldots,16$ 。在十维空间中，洛伦兹群有两个不等价的旋量表示，分别记为外尔和反外尔。它们通过旋量指标的位置来区分，外尔旋量指标在上方 Ψ^α ，反外尔旋量指标在下方 χ_α ，且不能进行升降。伽马矩阵 $\gamma^m_{\alpha\beta}$ 和 $\gamma_m^{\alpha\beta}$ 是 $SO(9,1)$ 克利福德代数 $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{I}_{32\times 32}$ 的 32×32 狄拉克矩阵 Γ^m 的 16×16 非对角对称泡利矩阵。它们满足 $\gamma^m_{\alpha\beta}(\gamma^n)^{\beta\rho} + \gamma^n_{\alpha\beta}(\gamma^m)^{\beta\rho} = 2\eta^{mn}\delta^\rho_\alpha$ 。

The nonlinear equations of motion following from the above constraint have linearized counterparts written in terms of linearized superfield connections $A_\alpha(x,\theta)$, $A^m(x,\theta)$ and their field strengths $W^\alpha(x,\theta)$, and $F^{mn}(x,\theta)$:

由上述约束得到的非线性运动方程存在线性化形式，用线性化超场联络 $A_\alpha(x,\theta)$ 、$A^m(x,\theta)$ 及其场强 $W^\alpha(x,\theta)$ 和 $F^{mn}(x,\theta)$ 写为:

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta}A_m,\ D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$

$$D_\alpha F_{mn} = \partial_{[m}\big(\gamma_{n]}W\big)_\alpha,\ D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}. \tag{2}$$

These linearized superfields will enter the expressions for the massless vertex operators of the pure spinor formalism and will be the main actors in the composition of pure spinor superspace expressions to be reviewed below. In this context, it is essential to know how these superfields are expanded in a series of θ variables.

这些线性化超场会出现在纯自旋形式论的无质量顶点算子表达式中，也是下文将要回顾的纯自旋超空间表达式构造中的核心对象。在此背景下，必须了解这些超场如何按 θ 变量级数展开。

The linearized superfields can be expanded in the so-called Harnad-Shnider gauge $\theta^\alpha A_\alpha(x,\theta) = 0$ in terms of the gluon e^m_i and gluino χ^α_i polarizations of a particle state labeled by i [78]. For convenience we strip off the universal plane-wave factor $e^{k\cdot x}$ that carries all the x dependence from the superfields and define their θ -dependent factor as $A^i_\alpha(x,\theta) = A^i_\alpha(\theta)\, e^{k\cdot x}$, etc. One can show that

线性化超场可按照所谓的哈纳德-施奈德规范 $\theta^\alpha A_\alpha(x,\theta) = 0$ ，以 i 标记粒子态的胶子 e^m_i 和胶微子 χ^α_i 极化展开 [78]。为方便起见，我们从超场中剥离承载所有 x 依赖的通用平面波因子 $e^{k\cdot x}$ ，将其依赖 θ 的因子定义为 $A^i_\alpha(x,\theta) = A^i_\alpha(\theta)\, e^{k\cdot x}$ ，依此类推。可以证明

$$A^i_\alpha(\theta) = \frac{1}{2}(\theta\gamma_m)_\alpha e^m_i + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m\chi_i) - \frac{1}{32}(\theta\gamma_m)^\alpha(\theta\gamma^{mnp}\theta)\, f^i_{np} \tag{3}$$

$$+\frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)\, k^i_n\left(\chi^i\gamma_p\theta\right) + \frac{1}{1152}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma^{pqr}\theta)$$

$$k^n_i f^{qr}_i + \ldots$$

$$A^m_i(\theta) = e^m_i + (\theta\gamma^m\chi_i) - \frac{1}{8}(\theta\gamma^{mpq}\theta)\, f^{pq}_i + \frac{1}{12}(\theta\gamma^{mnp}\theta)\, k^n_i\,(\chi_i\gamma^p\theta)$$

$$+\frac{1}{192}\left(\theta\gamma^{m}{}_{nr}\theta\right)\left(\theta\gamma^{r}{}_{pq}\theta\right)k_i^n f_i^{pq}-\frac{1}{480}\left(\theta\gamma^{m}{}_{nr}\theta\right)\left(\theta\gamma^{r}{}_{pq}\theta\right)k_i^n k_i^p$$

$$\left(\chi_i\gamma^q\theta\right)+\ldots$$

$$W_i^\alpha(\theta)=\chi_i^\alpha+\frac{1}{4}\left(\theta\gamma^{mn}\right)^\alpha f_{mn}^i-\frac{1}{4}\left(\theta\gamma_{mn}\right)^\alpha k_i^m\left(\chi_i\gamma^n\theta\right)-\frac{1}{48}\left(\theta\gamma_m{}^q\right)^\alpha$$

$$\left(\theta\gamma_{qnp}\theta\right)k_i^m f_i^{np}$$

$$+\frac{1}{96}\left(\theta\gamma_m{}^q\right)^\alpha\left(\theta\gamma_{qnp}\theta\right)k_i^m k_i^n\left(\chi_i\gamma^p\theta\right)-\frac{1}{1920}\left(\theta\gamma_m{}^r\right)^\alpha\left(\theta\gamma_{nr}{}^s\theta\right)\left(\theta\gamma_{spq}\theta\right)$$

$$k_i^m k_i^n f_i^{pq}+\ldots$$

$$F_i^{mn}(\theta)=f_i^{mn}-k_i^{[m}\left(\chi_i\gamma^{n]}\theta\right)+\frac{1}{8}\left(\theta\gamma_{pq}{}^{[m}\theta\right)k_i^{n]}f_i^{pq}-\frac{1}{12}\left(\theta\gamma_{pq}{}^{[m}\theta\right)k_i^{n]}k_i^p\left(\chi_i\gamma^q\theta\right)$$

$$-\frac{1}{192}\left(\theta\gamma_{ps}{}^{[m}\theta\right)k_i^{n]}k_i^p f_i^{qr}\left(\theta\gamma_{qr}^s\theta\right)+\frac{1}{480}\left(\theta\gamma^{[m}{}_{ps}\theta\right)k_i^{n]}k_i^p k_i^q$$

$$\left(\chi_i\gamma^r\theta\right)\left(\theta\gamma^s{}_{qr}\theta\right)+\ldots,$$

where $f_i^{mn}=k^m e_i^n-k^n e_i^m$ is the linearized field strength of the i th gluon and the terms in the ellipsis of order $\theta^{>5}$ will not contribute in pure spinor superspace expressions to be reviewed below.

其中 $f_i^{mn}=k^m e_i^n-k^n e_i^m$ 是第 i 个胶子的线性化场强，省略号中 $\theta^{>5}$ 阶的项不会对下文将要回顾的纯旋量超空间表达式产生贡献。

Non-minimal Pure Spinor Formalism

非最小纯旋子形式

It is customary to distinguish two very closely related versions of the pure spinor formalism: minimal [25] and non-minimal [22]. Both are based on the ideas of [1], but the non-minimal incarnation introduces new variables on the worldsheet and admits a simpler ”topological” multiloop amplitude prescription.

通常我们将纯旋子形式区分为两种联系紧密的版本: 最小版 [25] 和非最小版 [22]。二者均基于文献 [1] 的思想，但非最小版本在世界面上引入了新变量，并且拥有更简单的“拓扑”多圈振幅表述。

The left-moving sector of the non-minimal pure spinor formalism is composed of the fields ∂x^m, p_α, w_α, s^α of conformal weight one and of θ^α, λ^α, $\overline{\lambda}_\alpha$, r_α of conformal weight zero, where $m=0,1\ldots,9$, and $\alpha=1,\ldots,16$ are the vector and spinorial indices of $SO(10)$. The worldsheet action is

非最小纯旋子形式的左动 sector 由共形权重为 1 的场 $\partial x^m, p_\alpha, w_\alpha, s^\alpha$ 和共形权重为 0 的场 $\theta^\alpha, \lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha$ 构成，其中 $m = 0, 1 \ldots, 9$ ，$\alpha = 1, \ldots, 16$ 是 $SO(10)$ 的矢量和旋量指标。世界面作用量为

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \left(\partial x^m \overline{\partial} x_m + \alpha' p_\alpha \overline{\partial} \theta^\alpha - \alpha' w_\alpha \overline{\partial} \lambda^\alpha - \alpha' \bar{w}^\alpha \overline{\partial} \lambda_\alpha + \alpha' s^\alpha \overline{\partial} r_\alpha \right), \tag{4}$$

and α' denotes the inverse string tension. The field λ^α is bosonic and satisfies the pure spinor constraint:

且 α' 表示逆弦张力。场 λ^α 是玻色场，满足纯旋子约束:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{5}$$

The field $\overline{\lambda}_\alpha$ is bosonic while r_α is fermionic, and they satisfy the constraints:

场 $\overline{\lambda}_\alpha$ 是玻色场，r_α 是费米场，二者满足约束:

$$\overline{\lambda}_\alpha \gamma_m^{\alpha\beta} \overline{\lambda}_\beta = 0, \ \overline{\lambda}_\alpha \gamma_m^{\alpha\beta} r_\beta = 0. \tag{6}$$

The OPEs of the matter variables are given by

物质变量的算符乘积展开为

$$X^m(z, \bar{z}) X_n(w, \bar{w}) \sim -\frac{\alpha'}{2} \delta^m_n \ln |z - w|^2, \ p_\alpha(z) \theta^\beta(w) \sim \frac{\delta^\beta_\alpha}{z - w}, \tag{7}$$

while the OPEs of the ghost variables do not follow from a free-field calculation due to the constraints above. In certain circumstances, however, the variables $(w_\alpha, \lambda^\alpha)$ can be viewed as a conjugate pair with canonical OPE. The Green-Schwarz constraint $d_\alpha(z)$ and the supersymmetric momentum $\Pi^m(z)$ have conformal weight +1 and are given by

由于上述约束，鬼变量的算符乘积展开无法通过自由场计算得到。但在某些条件下，变量 $(w_\alpha, \lambda^\alpha)$ 可被视为具有正则算符乘积展开的共轭对。格林-施瓦茨约束 $d_\alpha(z)$ 和超对称动量 $\Pi^m(z)$ 的共形权重为 +1，表达式为

$$d_\alpha(z) = p_\alpha - \frac{1}{\alpha'} (\gamma^m \theta)_\alpha \partial x_m - \frac{1}{4\alpha'} (\gamma^m \theta)_\alpha (\theta \gamma_m \partial \theta), \tag{8}$$

$$\Pi^m(z) = \partial x^m + \frac{1}{2} (\theta \gamma^m \partial \theta).$$

These fields satisfy the following OPEs:

这些场满足如下算符乘积展开:

$$d_\alpha(z) d_\beta(w) \sim -\frac{\gamma^m_{\alpha\beta} \Pi_m}{z - w}, \ \Pi^m(z) \Pi^n(w) \sim -\frac{\eta^{mn}}{(z - w)^2}, \tag{9}$$

$$d_\alpha(z)\,\Pi^m(w) \sim \frac{(\gamma^m \partial\theta)_\alpha}{z-w},$$

In addition, if $f(x(w), \theta(w))$ does not depend on derivatives $\partial^k x$ nor $\partial^k \theta$

> 此外，如果 $f(x(w), \theta(w))$ 不依赖于导数 $\partial^k x$ 也不依赖于 $\partial^k \theta$

$$d_\alpha(z) f(x(w), \theta(w)) \sim \frac{D_\alpha f}{z-w},\ \Pi^m(z) f(x(w), \theta(w)) \sim -\frac{k^m f}{z-w} \tag{10}$$

The non-minimal BRST charge

> 非最小 BRST 荷

$$Q = \oint (\lambda^\alpha d_\alpha + \bar{w}^\alpha r_\alpha), \tag{11}$$

can be shown to be nilpotent $Q^2 = 0$ using the OPEs (8) and the pure spinor constraint (5). Physical states are required to be in the cohomology of (11), and it will be shown below that the cohomology is independent on the quartet of nonminimal variables $\left(\bar{w}^\alpha, \bar{\lambda}_\alpha, s^\alpha, r_\alpha\right)$.

> 利用式 (8) 的算符乘积展开和式 (5) 的纯旋子约束可以证明其幂零 $Q^2 = 0$ 。物理态要求属于式 (11) 的上同调，下文将证明该上同调与非最小变量四元组 $\left(\bar{w}^\alpha, \bar{\lambda}_\alpha, s^\alpha, r_\alpha\right)$ 无关。

The constraint (5) implies that the conjugate momentum w_α to the pure spinor λ^α can only appear in gauge-invariant combinations under

> 约束 (5) 表明，纯旋子 λ^α 的共轭动量 w_α 只能出现在满足下述规范变换的规范不变组合中

$$\delta w_\alpha(z) = \Omega_m(z)(\gamma^m \lambda)_\alpha. \tag{12}$$

The basic gauge-invariant quantities are the current J_λ , the energy momentum tensor T_λ , and the Lorentz current N_{mn} given by

> 基本规范不变量是流 J_λ 、能量动量张量 T_λ 和洛伦兹流 N_{mn} ，表达式为

$$J_\lambda(z) = w_\alpha \lambda^\alpha,\ T_\lambda(z) = w_\alpha \partial \lambda^\alpha,\ N^{mn}(z) = \frac{1}{2}(w \gamma^{mn} \lambda). \tag{13}$$

Since the conjugate pair $(\lambda^\alpha, w_\alpha)$ is not free due to the pure spinor constraint, the OPEs of these gauge invariants are computed using the $U(5)$ parameterization of λ^α , with the $SO(10)$ -covariant result [1]:

> 由于纯旋子约束，共轭对 $(\lambda^\alpha, w_\alpha)$ 不是自由的，因此这些规范不变量的算符乘积展开需要通过 λ^α 的 $U(5)$ 参数化计算，得到 $SO(10)$ 协变的结果 [1]:

$$N^{mn}(z)\lambda^\alpha(w) \sim \frac{\frac{1}{2}(\gamma^{mn}\lambda)^\alpha(w)}{z-w},\ J_\lambda(z)\lambda^\alpha(w) \sim \frac{\lambda^\alpha(w)}{z-w}, \tag{14}$$

$$N^{mn}(z)J_\lambda(w) \sim \text{ regular}, \; J_\lambda(z)J_\lambda(w) \sim \frac{-4}{(z-w)^2},$$

$$N_{mn}(z)T_\lambda(w) \sim \frac{N_{mn}(w)}{(z-w)}, \; J_\lambda(z)T_\lambda(w) \sim \frac{-8}{(z-w)^3} + \frac{J_\lambda(w)}{(z-w)^2},$$

$$N^{mn}(z)N_{pq}(w) \sim \frac{N^m{}_p\delta^n_q - N^n{}_p\delta^m_q + N^n{}_q\delta^m_p - N^m{}_q\delta^n_p}{z-w} - \frac{3\left(\delta^n_p\delta^m_q - \delta^n_q\delta^m_p\right)}{(z-w)^2},$$

$$T_\lambda(z)T_\lambda(w) \sim \frac{11}{(z-w)^4} + \frac{2T_\lambda(w)}{(z-w)^2} + \frac{\partial T_\lambda(w)}{z-w}.$$

Similarly, the constraints (6) imply that the conjugates $\bar{w}^\alpha$ and s^α of conformal weight +1 can only appear in gauge-invariant combinations under

类似地，约束 (6) 表明，共形权重为 +1 的共轭量 $\bar{w}^\alpha$ 和 s^α 只能出现在满足下述规范变换的规范不变组合中

$$\delta\bar{w}^\alpha = \overline{\Omega}_m\left(\gamma^m\bar{\lambda}\right)^\alpha - \phi_m(\gamma^m r)^\alpha, \; \delta s^\alpha = \phi_m\left(\gamma^m\bar{\lambda}\right)^\alpha, \tag{15}$$

where $\overline{\Omega}_m$ and ϕ_m are arbitrary parameters. The non-minimal counterparts of the gauge invariants (13) are given by [22]

其中 $\overline{\Omega}_m$ 和 ϕ_m 是任意参数。式 (13) 中规范不变量对应的非最小版本由文献 [22] 给出如下

$$\bar{N}_{mn} = \frac{1}{2}\left(\bar{w}\gamma_{mn}\bar{\lambda} - s\gamma_{mn}r\right), \; J_{\bar{\lambda}} = \bar{w}^\alpha\bar{\lambda}_\alpha - s^\alpha r_\alpha, \; T_{\bar{\lambda}} = \bar{w}^\alpha\partial\bar{\lambda}_\alpha - s^\alpha\partial r_\alpha,$$

(16)

with additional gauge invariants

带有额外规范不变量

$$S_{mn} = \frac{1}{2}s\gamma_{mn}\bar{\lambda}, \; S = s^\alpha\bar{\lambda}_\alpha. \tag{17}$$

The above gauge invariants are related via the BRST charge:

上述规范不变量通过 BRST 电荷关联:

$$\bar{N}_{mn} = QS_{mn}, \; J_{\bar{\lambda}} = QS, \; T_{\bar{\lambda}} = Q\left(s^\alpha\partial\bar{\lambda}_\alpha\right), \tag{18}$$

Therefore, the operator $\bar{q} = \oint J_{\bar{\lambda}}$ counting the non-minimal variables is BRST exact, and satisfies $\bar{q}\bar{\lambda}_\alpha = \bar{\lambda}_\alpha$ and $\bar{q}r_\alpha = r_\alpha$. Therefore if a BRST-closed state $Q\Psi = 0$ has nonvanishing non-minimal $\bar{q}$ charge $\bar{q}\Psi = n\Psi$ with $n \neq 0$, it is also BRST-exact; $\Psi = \frac{\bar{q}}{n}\Psi$. And since S and S^{mn} are not closed and $\left(\bar{N}_{mn}, J_{\bar{\lambda}}, T_{\bar{\lambda}}\right)$ are exact, the quartet $\left(\bar{w}^\alpha, \bar{\lambda}_\alpha, s^\alpha, r_\alpha\right)$ of non-minimal variables decouples from the cohomology in what is known as the Kugo-Ojima quartet mechanism.

因此，计数非极小变量的算符 $\bar{q} = \oint \bar{J}_{\bar{\lambda}}$ 是 BRST 恰当的，且满足 $\bar{q}\bar{\lambda}_\alpha = \bar{\lambda}_\alpha$ 和 $\bar{q}r_\alpha = r_\alpha$ 。因此若 BRST 闭态 $Q\Psi = 0$ 具有非零的非极小 $\bar{q}$ 电荷 $\bar{q}\Psi = n\Psi$ 且满足 $n \neq 0$ ，则该态也是 BRST 恰当的；$\Psi = \frac{\bar{q}}{n}\Psi$ 。又由于 S 和 S^{mn} 不闭而 $\left(\bar{N}_{mn}, \bar{J}_{\bar{\lambda}}, T_{\bar{\lambda}}\right)$ 恰当，非极小变量的四重态 $\left(\bar{w}^\alpha, \bar{\lambda}_\alpha, s^\alpha, r_\alpha\right)$ 退耦从上同调中分离，这就是著名的久保-尾岛四重态机制。

Moreover, the energy-momentum tensor

此外，能量动量张量

$$T(z) = -\frac{1}{\alpha'}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + w_\alpha \partial\lambda^\alpha + \bar{w}^\alpha \partial\bar{\lambda}_\alpha - s^\alpha \partial r_\alpha, \tag{19}$$

is related to the BRST charge through the b ghost as $\{Q, b(z)\} = T(z)$, where

通过 b 鬼与 BRST 电荷关联为 $\{Q, b(z)\} = T(z)$ ，其中

$$\begin{gathered}
b = s^\alpha \partial\bar{\lambda}_\alpha + \frac{1}{4\left(\lambda\bar{\lambda}\right)}\left[2\Pi^m\left(\bar{\lambda}\gamma_m d\right) - N_{mn}\left(\bar{\lambda}\gamma^{mn}\partial\theta\right) - J_\lambda\left(\bar{\lambda}\partial\theta\right) - \left(\bar{\lambda}\partial^2\theta\right)\right] \\
+\frac{\left(\bar{\lambda}\gamma^{mnp}r\right)}{192\left(\lambda\bar{\lambda}\right)^2}\left[\frac{\alpha'}{2}\left(d\gamma_{mnp}d\right) + 24N_{mn}\Pi_p\right] \\
-\frac{\alpha'}{2}\frac{\left(r\gamma_{mnp}r\right)\left(\bar{\lambda}\gamma^m d\right)N^{np}}{16\left(\lambda\bar{\lambda}\right)^3} + \frac{\alpha'}{2}\frac{\left(r\gamma_{mnp}r\right)\left(\bar{\lambda}\gamma^{pqr}r\right)N^{mn}N_{qr}}{128\left(\lambda\bar{\lambda}\right)^4}.
\end{gathered}$$

(20)

After extracting the non-minimal $U(1)$ ghost-number current

提取出非极小 $U(1)$ 鬼数流后

$$J(z) = w_\alpha\lambda^\alpha - s^\alpha r_\alpha - \frac{2\left(\left(\bar{\lambda}\partial\lambda\right) + \left(r\partial\theta\right)\right)}{\left(\lambda\bar{\lambda}\right)} + \frac{2\left(\lambda r\right)\left(\bar{\lambda}\partial\theta\right)}{\left(\lambda\bar{\lambda}\right)^2} \tag{21}$$

from the double pole of the b ghost with the integrand $\lambda^\alpha d_\alpha + \bar{w}^\alpha r_\alpha$ of the BRST charge, the non-minimal pure spinor formalism was shown in [22] to be a critical $N = 2$ topological string. More precisely, using the terminology $G^+(z) = \left(\lambda^\alpha d_\alpha + \overline{\omega}^\alpha r_\alpha\right), G^-(z) = b(z)$, one can show that $T(z), G^+(z), G^-(z)$ and $J(z)$ satisfy the OPEs [22, 83] :

从 BRST 电荷被积函数 $\lambda^\alpha d_\alpha + \bar{w}^\alpha r_\alpha$ 与 b 鬼的双极点中分离出该流后，文献 [22] 证明非极小纯旋量形式是临界 $N = 2$ 拓扑弦。更准确地说，采用术语 $G^+(z) = \left(\lambda^\alpha d_\alpha + \overline{\omega}^\alpha r_\alpha\right), G^-(z) = b(z)$ 可以证明 $T(z), G^+(z), G^-(z)$ 和 $J(z)$ 满足 OPE 关系 [22, 83] :

$$T(z)T(w) \sim \frac{2T}{(z-w)^2} + \frac{\partial T}{(z-w)} \tag{22}$$

$$T(z)G^+(w) \sim \frac{G^+}{(z-w)^2} + \frac{\partial G^+}{(z-w)}$$

$$T(z)\,G^{-}(w) \sim \frac{2G^{-}}{(z-w)^{2}} + \frac{\partial G^{-}}{(z-w)}$$

$$G^{+}(z)\,G^{-}(w) \sim \frac{3}{(z-w)^{3}} + \frac{J}{(z-w)^{2}} + \frac{T}{(z-w)}$$

$$T(z)\,J(w) \sim \frac{-3}{(z-w)^{3}} + \frac{J}{(z-w)^{2}} + \frac{\partial J}{(z-w)}$$

$$J(z)\,G^{\pm}(w) \sim \pm\frac{G^{\pm}}{(z-w)}$$

$$J(z)\,J(w) \sim \frac{3}{(z-w)^{2}}$$

$$G^{\pm}(z)\,G^{\pm}(w) \sim \text{ regular,}$$

which identifies them as the generators of a $\hat{c} = 3N = 2$ twisted topological conformal algebra. As such, not only the BRST charge has to be nilpotent but also the b ghost (see, e.g.,[85]). A proof that $b^2 = 0$ with (20) can be found in [83, 84]. Note that the simpler BRST-equivalent $U(1)$ charge $J(z) = w_\alpha \lambda^\alpha - \bar{w}^\alpha \overline{\lambda}_\alpha$ was shown in [22] to preserve the essential features of the topological string and therefore can be used instead of (21) to define the ghost number of pure spinor operators.

由此可确认它们是 $\hat{c} = 3N = 2$ 扭变拓扑共形代数的生成元。据此，不仅 BRST 电荷必须是幂零的，b 鬼也必须幂零 (参见例如文献 [85])。文献 [83, 84] 中可找到关于 $b^2 = 0$ 结合 (20) 式的证明。注意，文献 [22] 已证明更简单的 BRST 等价 $U(1)$ 电荷 $J(z) = w_\alpha \lambda^\alpha - \bar{w}^\alpha \overline{\lambda}_\alpha$ 保留了拓扑弦的核心性质，因此可替代 (21) 式定义纯旋量算符的鬼数。

Vertex Operators and Amplitude Prescription

顶点算子与振幅 prescriptions

Vertex operators for massless open-string states are constructed from the linearized SYM superfields of (2) as

无质量开弦态的顶点算子由 (2) 的线性化 SYM 超场构造如下

$$V = \lambda^\alpha A_\alpha(x,\theta)\,, \tag{23}$$

$$U = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + \frac{1}{2} N_{mn} F^{mn}(x,\theta)$$

and are independent on the non-minimal variables using the quartet mechanism discussed above. V is called the unintegrated vertex and has conformal weight zero, and U is called the integrated vertex and has conformal weight $+1$. They are related via the BRST charge (11) by $QU = \partial V$, so the integrated vertex is BRST closed up to a total derivative on the worldsheet. The unintegrated vertex is BRST closed as a consequence of (10), the equation of motion (2), as well as the pure spinor constraint (5):

利用前文讨论的四重态机制，它们不依赖于非最小变量。V 称为非积分顶点，共形权重为零；U 称为积分顶点，共形权重为 +1。它们通过 BRST 荷 (11) 由 $QU = \partial V$ 关联，因此积分顶点在世界面全导数范围内是 BRST 闭的。利用 (10)、运动方程 (2) 以及纯旋量约束 (5)，可得非积分顶点是 BRST 闭的:

$$QV = \lambda^\alpha \lambda^\beta D_\alpha A_\beta = \frac{1}{2} \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta A_m = 0. \tag{24}$$

Their closed-string versions are obtained by a double copy of the open-string vertex operators with the plane-wave factor stripped off, that is, $|V|^2 = V(\theta) \widetilde{V}(\tilde{\theta}) e^{k\cdot x}$ where $V(\theta) = \lambda^\alpha A_\alpha(\theta)$ with $A_\alpha(\theta)$ as in (3), and similarly for $|U|^2$.

闭弦版本可由开弦顶点算子去掉平面波因子后做双拷贝得到,即 $|V|^2 = V(\theta) \widetilde{V}(\tilde{\theta}) e^{k\cdot x}$,其中 $V(\theta) = \lambda^\alpha A_\alpha(\theta)$ 满足 $A_\alpha(\theta)$ ，与 (3) 中形式一致，$|U|^2$ 同理。

The prescription to calculate n -point closed-string amplitudes at genus g is

亏格为 g 的 n 点闭弦振幅的计算规则为

$$\mathcal{A}_{g=0} = \kappa^n e^{-2\lambda} \int_{\sum} \prod_{j=2}^{n-2} d^2 z_j \left| \langle \mathcal{N}_0 V_1(0) U_j(z_j) V_{n-1}(1) V_n(\infty) \rangle \right|^2 \tag{25}$$

$$\mathcal{A}_{g=1} = \frac{1}{2} \kappa^n \int_{\sum,\mathcal{M}_1} d^2 \tau_1 \prod_{j=2}^{n} d^2 z_j \left| \langle \mathcal{N}(b,\mu_1) V_1(0) U_j(z_j) \rangle \right|^2 \tag{26}$$

$$\mathcal{A}_{g>1} = \kappa^n e^{2\lambda} \left(1 - \frac{1}{2}\delta_{g,2}\right) \int_{\sum,\mathcal{M}_g} d^{3g-3}\tau \prod_{j=1}^{n} d^2 z_j \left| \left\langle U_j(z_j) \prod_{I=1}^{3g-3} (b,\mu_I) \mathcal{N} \right\rangle \right|^2$$

(27)

where $U(z)$ is the integrated vertex operator (23), τ_I for $I = 1,\dots,3g-3$ are the complex Teichmüller parameters with μ_I their associated Beltrami differentials, the b ghost is given by (20) and

其中 $U(z)$ 是积分顶点算子 (23)，对应 $I = 1,\dots,3g-3$ 的 τ_I 是复泰希米勒参数，μ_I 是其对应的贝尔特拉米微分，b 鬼由式 (20) 给出，且

$$(b,\mu_I) = \frac{1}{2\pi} \int d^2 z b_{zz} \mu_I^{\bar{z}} \bar{z}, \tag{28}$$

$\mathcal{N}$ is the regularization factor (36) responsible for convergence as $(\lambda\bar{\lambda}) \to \infty$, κ is the normalization of the vertex operators ($\kappa^2 = e^{2\lambda}\pi/\alpha'^2$ by unitarity) and $e^{2(g-1)\lambda}$ is the string coupling constant as in [43]. The factor of 1/2 in the genus-two amplitude is required because all genus-two curves have a $\mathbb{Z}_2$ symmetry [86]. In addition, $|.|^2$ signifies the holomorphic square of the integrand with the plane waves of the vertex operators dealt with as described above, and it is important to emphasize that all calculations are done in the left- and right-moving sectors separately using the chiral-splitting formalism explained below.

$\mathcal{N}$ 是保证 $\left(\lambda\bar{\lambda}\right) \rightarrow \infty$ 区域收敛的正规化因子 (36)，κ 是顶点算子通过幺正性得到的归一化 ($\kappa^2 = e^{2\lambda}\pi/\alpha'^2$ ，$e^{2(g-1)\lambda}$ 是 [43] 中定义的弦耦合常数。亏格 2 振幅中需要引入因子 1/2 ，原因是所有亏格 2 曲线都存在 $\mathbb{Z}_2$ 对称性 [86]。此外，$|.|^2$ 表示被积函数的全纯平方，其中顶点算子的平面波按前文所述处理，需要强调的是，所有计算都利用后文说明的手征分裂形式，在左行和右行区域分别进行。

Integration of nonzero modes The OPEs in a genus g Riemann surface are used to integrate out the nonzero modes of the fields of conformal weight $+1$. To do this, we first separate off the zero modes as (using $d_\alpha(z)$ to illustrate the procedure)

非零模积分我们利用亏格 g 黎曼面上的算子乘积展开对共形权重 +1 场的非零模做积分。具体来说，我们首先分离出零模，以 $d_\alpha(z)$ 为例说明步骤:

$$d_\alpha(z) = \widehat{d}_\alpha(z) + \sum_{I=1}^{g} d_\alpha^I \omega_I(z), \oint_{A_I} \widehat{d}_\alpha = 0 \tag{29}$$

where $\omega_I(z)$ are g holomorphic one-forms satisfying $\oint_{A_I} \omega_J(z)\, dz = \delta_{IJ}$ and A_I represents the A cycles of the Riemann surface. Then, the nonzero modes (indicated by hats) are integrated out via their OPEs. For example,

其中 $\omega_I(z)$ 是满足 $\oint_{A_I} \omega_J(z)\, dz = \delta_{IJ}$ 的 g 个全纯一维形式，A_I 代表黎曼面的 A 个闭链。随后，非零模 (以 hat 标记) 通过其 OPE 积分掉。例如:

$$\widehat{p}_\alpha(z)\theta^\beta(y) \sim \partial_z \ln E(z,y)\,\delta_\alpha^\beta$$

$$\widehat{d}_\alpha(z) K(x(y), \theta(y)) \sim \partial_z \ln E(z,y) D_\alpha K(x(y), \theta(y)) \tag{30}$$

$$\widehat{\Pi}_m(z) K(x(y), \theta(y)) \sim -\partial_z \ln E(z,y) \partial_m K(x(y), \theta(y))$$

where $E(z,y)$ is the prime form and $K(x,\theta)$ is an arbitrary superfield depending on x and θ , but not on the worldsheet derivatives of these fields. In the limit where $z \to y$, the prime form behaves as $E(z,y) \sim z-y$, and the propagator $\partial_z \ln E(z,y)$ displays its distinctive singular structure $\sim 1/(z-y)$ seen in (8). The OPE of the $x^m(z,\bar{z})$ fields

其中 $E(z,y)$ 是素形式，$K(x,\theta)$ 是依赖于 x 和 θ 的任意超场，不依赖于这些场的世界面导数。在 $z \to y$ 极限下，素形式满足 $E(z,y) \sim z-y$ ，传播子 $\partial_z \ln E(z,y)$ 呈现出 (8) 中可见的特征奇异结构 $\sim 1/(z-y)$ 。$x^m(z,\bar{z})$ 场的 OPE 为

$$X^m(z,\bar{z}) X_n(w,\bar{w}) \sim -\frac{\alpha'}{2}\delta_n^m G(z,w), \tag{31}$$

with $G(z,w)$ the genus- g Green function, couples the left- and right-movers and motivates the chiral splitting techniques developed by D'Hoker and Phong.

其中 $G(z,w)$ 是亏格- g 格林函数，它耦合左动模与右动模，推动了 D'Hoker 和 Phong 发展的手征分裂技术。

Zero-mode integrations The zero-mode integrations that remain after integrating out the nonzero modes via OPEs are performed using

零模积分通过 OPE 积去非零模后剩余的零模积分，利用下式完成

$$\langle \ldots \rangle = \int [d\theta]\,[dr]\,[d\lambda]\left[d\bar{\lambda}\right]\prod_{I=1}^{g}\left[dd^I\right]\left[ds^I\right]\left[d\bar{w}^I\right]\left[dw^I\right]\ldots \tag{32}$$

where [45]

其中文献 [45]

$$[d\lambda]\,T_{\alpha_1\ldots\alpha_5} = c_\lambda\left(\varepsilon\cdot d^{11}\lambda\right)_{\alpha_1\ldots\alpha_5},\ [dw] = c_w\left(T\cdot\varepsilon\cdot d^{11}w\right)$$

$$\left[d\bar{\lambda}\right]\bar{T}^{\alpha_1\ldots\alpha_5} = c_{\bar{\lambda}}\left(\varepsilon\cdot d^{11}\bar{\lambda}\right)^{\alpha_1\ldots\alpha_5},\ [dr] = c_r\left(\bar{T}\cdot\varepsilon\cdot\partial_r^{11}\right)$$

$$[d\bar{w}]\,T_{\alpha_1\ldots\alpha_5} = c_{\bar{w}}\left(\varepsilon\cdot d^{11}\bar{w}\right)_{\alpha_1\ldots\alpha_5}\ \left[ds^I\right] = c_s\left(T\cdot\varepsilon\cdot\partial_{s^I}^{11}\right)$$

$$[d\theta] = c_\theta d^{16}\theta\ \left[dd^I\right] = c_d d^{16}d^I. \tag{33}$$

with the shorthand $\left(\varepsilon\cdot d^{11}\lambda\right)_{\alpha_1\ldots\alpha_5} := \frac{1}{11!}\varepsilon_{\alpha_1\ldots\alpha_{16}}d\lambda^{\alpha_6}\ldots d\lambda^{\alpha_{16}}$, and its contraction $\left(\bar{T}\cdot\varepsilon\cdot d^{11}\lambda\right) = \frac{1}{11!5!}\bar{T}^{\alpha_1\ldots\alpha_5}\left(\varepsilon\cdot d^{11}\lambda\right)_{\alpha_1\ldots\alpha_5}$ with similar expressions for the others. The expressions of $T_{\alpha_1\ldots\alpha_5}$ and $\bar{T}^{\alpha_1\ldots\alpha_5}$ are given by

这里采用简写 $\left(\varepsilon\cdot d^{11}\lambda\right)_{\alpha_1\ldots\alpha_5} := \frac{1}{11!}\varepsilon_{\alpha_1\ldots\alpha_{16}}d\lambda^{\alpha_6}\ldots d\lambda^{\alpha_{16}}$，它的缩并为 $\left(\bar{T}\cdot\varepsilon\cdot d^{11}\lambda\right) = \frac{1}{11!5!}\bar{T}^{\alpha_1\ldots\alpha_5}\left(\varepsilon\cdot d^{11}\lambda\right)_{\alpha_1\ldots\alpha_5}$，其余量有类似表达式。$T_{\alpha_1\ldots\alpha_5}$ 和 $\bar{T}^{\alpha_1\ldots\alpha_5}$ 的表达式由下式给出

$$T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}\left(\gamma_{mnp}\right)_{\alpha_4\alpha_5} \tag{34}$$

$$\bar{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \left(\bar{\lambda}\gamma^m\right)^{\alpha_1}\left(\bar{\lambda}\gamma^n\right)^{\alpha_2}\left(\bar{\lambda}\gamma^p\right)^{\alpha_3}\left(\gamma_{mnp}\right)^{\alpha_4\alpha_5}$$

and they can be shown to be totally antisymmetric due to the pure spinor constraints and satisfy $T\cdot\bar{T} = 5!2^6\left(\lambda\bar{\lambda}\right)^3$. Finally, the normalizations are given by

可以证明，它们受纯旋量约束作用是全反对称的，并且满足 $T\cdot\bar{T} = 5!2^6\left(\lambda\bar{\lambda}\right)^3$。最后，归一化因子由下式给出

$$c_\lambda = \left(\frac{\alpha'}{2}\right)^{-2}\left(\frac{A_g}{4\pi^2}\right)^{11/2}\ c_w = \left(\frac{\alpha'}{2}\right)^2\frac{1}{(2\pi)^{11}Z_g^{11/g}} \tag{35}$$

$$c_{\overline{\lambda}} = 2^6\left(\frac{\alpha'}{2}\right)^2\left(\frac{A_g}{4\pi^2}\right)^{11/2} \quad c_r = R\left(\frac{\alpha'}{2}\right)^{-2}\left(\frac{2\pi}{A_g}\right)^{11/2}$$

$$c_{\bar{w}} = \left(\frac{\alpha'}{2}\right)^{-2}\frac{\left(\lambda\overline{\lambda}\right)^3}{(2\pi)^{11}}Z_g^{-11/g} \quad c_s = \left(\frac{\alpha'}{2}\right)^2\frac{(2\pi)^{11/2}}{2^6R\left(\lambda\overline{\lambda}\right)^3}Z_g^{11/g}$$

$$c_\theta = \left(\frac{\alpha'}{2}\right)^4\left(\frac{2\pi}{A_g}\right)^{16/2} \quad c_d = \left(\frac{\alpha'}{2}\right)^{-4}(2\pi)^{16/2}Z_g^{16/g},$$

where $A_g = \int d^2z\sqrt{g}$ is the area of the genus- g Riemann surface. Moreover, $Z_g = 1/\sqrt{\det(2\,\mathrm{Im}\,\Omega)}$ where Ω_{IJ} is the period matrix, and R is a free parameter that is used to choose the normalization of the three-point amplitude at genus zero (after which the normalization of all genus- gn -point amplitudes is fixed). As shown in [45], the closed-string amplitudes are independent on the area A_g because the number of bosonic and fermionic variables of conformal weight 0 is the same, and independent on the choice of normalization of the holomorphic one-forms because the number of bosonic and fermionic variables of conformal weight +1 is the same. The factor

其中 $A_g = \int d^2z\sqrt{g}$ 是亏格- g 黎曼曲面的面积。此外，$Z_g = 1/\sqrt{\det(2\,\mathrm{Im}\,\Omega)}$，其中 Ω_{IJ} 是周期矩阵，R 是自由参数，用于选择零亏格三点振幅的归一化 (归一化确定后，所有亏格- gn 点振幅的归一化都被固定)。如文献 [45] 所示，由于共形权重为 0 的玻色变量和费米变量数量相等，闭弦振幅与面积 A_g 无关；又由于共形权重为 +1 的玻色变量和费米变量数量相等，因此振幅也与全纯一元形式的归一化选择无关。该因子

$$\mathcal{N} = e^{-\left(\lambda\overline{\lambda}\right)-(r\theta)+\sum_{I=1}^{g}\left(s^Id^I\right)-\left(w^I\bar{w}^I\right)}. \tag{36}$$

regulates the zero-mode integrations over the non-compact spaces of the bosonic variables $\lambda^\alpha, \overline{\lambda}_\alpha$ and $w^\alpha, \bar{w}_\alpha$ as $\left(\lambda\overline{\lambda}\right) \to \infty$ and $(w\bar{w}) \to \infty$ in a manner explained in [22]. The formula for the integration over the pure spinor variables was found in [47] using techniques from algebraic geometry:

按照文献 [22] 所述的方式, 规范了玻色变量 $\lambda^\alpha, \overline{\lambda}_\alpha$ 和 $w^\alpha, \bar{w}_\alpha$ 在非紧空间上的零模积分, 对应 $\left(\lambda\overline{\lambda}\right) \to \infty$ 和 $(w\bar{w}) \to \infty$ 的情形。纯旋量变量的积分公式由文献 [47] 利用代数几何技术得到:

$$\int [d\lambda]\left[d\overline{\lambda}\right]\left(\lambda\overline{\lambda}\right)^n e^{-\left(\lambda\overline{\lambda}\right)} = \left(\frac{A_g}{2\pi}\right)^{11}\frac{\Gamma(8+n)}{7!60}, \tag{37}$$

where $\Gamma(x)$ is the gamma function. The b ghost (20) has factors of $1/\left(\lambda\overline{\lambda}\right)$ which are not regularized by the regulator (36) as $\left(\lambda\overline{\lambda}\right) \to 0$. It was shown in [22] that as long as the integrands diverge slower than $1/\left(\lambda^{8+3g}\overline{\lambda}^{11}\right)$, the amplitudes are still well-defined due to a compensating factor of $\lambda^{8+3g}\overline{\lambda}^{11}$ arising from $\langle\mathcal{N}\ldots\rangle$ in (32). As explained in [22], this issue is closely related to the existence of the operator:

其中 $\Gamma(x)$ 是伽马函数。b 鬼场 (20) 含有 $1/\left(\lambda\overline{\lambda}\right)$ 因子，当 $\left(\lambda\overline{\lambda}\right) \to 0$ 时，这些因子无法被正规化因子 (36) 正规化。文献 [22] 已证明，只要被积函数的发散速度慢于 $1/\left(\lambda^{8+3g}\overline{\lambda}^{11}\right)$，由于 (32) 式中 $\langle\mathcal{N}\ldots\rangle$ 会产生补偿因子 $\lambda^{8+3g}\overline{\lambda}^{11}$，振幅仍然是良定义的。如文献 [22] 所述，该问题与下述算符的存在密切相关:

$$\xi = \frac{(\lambda\theta)}{\left(\lambda\bar{\lambda}\right) + (r\theta)} = \frac{(\lambda\theta)}{\left(\lambda\bar{\lambda}\right)} \sum_{n=0}^{11} \left(\frac{(r\theta)}{\left(\lambda\bar{\lambda}\right)} \right)^n \tag{38}$$

where the Taylor expansion ends at $n = 11$ because there are only 11 degrees of freedom in r_α due to the constraint (6). This operator trivializes the cohomology as $Q\xi = 1$, but $\langle \mathcal{N}\xi \left(\lambda^3\theta^5\right)\rangle$ diverges faster than $1/\left(\lambda^{8+3g}\bar{\lambda}^{11}\right)$; therefore, if the integrands were allowed to diverge too fast, they would also be BRST-exact. Forbidding such pathological behavior restricts the amplitude prescription to contain at most three b ghosts or, in other words, up to genus two. By regularizing the b ghost to remove the singularity as $\left(\lambda\bar{\lambda}\right) \to 0$, an alternative prescription that allows amplitudes at arbitrary genus to be well-defined was proposed in [26].

其中泰勒展开在 $n = 11$ 处截断，这是因为约束条件 (6) 使得 r_α 中仅存在 11 个自由度。该算符将上同调平凡化为 $Q\xi = 1$ ，但 $\langle \mathcal{N}\xi \left(\lambda^3\theta^5\right)\rangle$ 的发散速度快于 $1/\left(\lambda^{8+3g}\bar{\lambda}^{11}\right)$ ；因此，如果允许被积函数发散过快，它们也会是 BRST 恰当的。禁止这类病态行为后，振幅方案最多只能包含三个 b 鬼，换句话说，亏格最多为二。通过正则化 b 鬼消除 $\left(\lambda\bar{\lambda}\right) \to 0$ 处的奇点，文献 [26] 提出了一种替代方案，可使任意亏格的振幅都得到良好定义。

As emphasized in [45], after the integration over $\left[dd^I\right]\left[ds^I\right]\left[dw^I\right]\left[d\bar{w}^I\right]\left[d\bar{w}^I\right]$ has been performed, the remaining integrations over $\lambda^\alpha, \bar{\lambda}_\beta, \theta^\delta$ and r_α are the same ones appearing in the prescription of the tree-level amplitudes, and therefore give rise to (non-minimal) pure spinor superspace expressions.

正如文献 [45] 所强调的，完成对 $\left[dd^I\right]\left[ds^I\right]\left[dw^I\right]\left[d\bar{w}^I\right]\left[d\bar{w}^I\right]$ 的积分后，剩余对 $\lambda^\alpha, \bar{\lambda}_\beta, \theta^\delta$ 和 r_α 的积分与树级振幅方案中出现的积分一致，因此得到 (非极小) 纯自旋 or 超空间表达式。

Chiral splitting To address the mixing of left- and right-movers via OPE contractions - an issue that prevents writing the closed-string correlator as an holomorphic square - the chiral-splitting procedure [38, 39, 75] introduces loop momenta ℓ_I^m

手征分裂为了解决 OPE 缩并带来的左动模与右动模混合问题——该问题导致无法将闭弦关联函数写为全纯平方——[38, 39, 75] 引入的手征分裂程序引入了圈动量 ℓ_I^m

$$\ell_I^m = \oint_{A_I} dz \Pi^m(z) : \tag{39}$$

in order to rewrite conformal correlators of the x^m -field in terms of an integral over ℓ_I . The integrand then becomes a product of left- and right-movers of schematic form $\mathcal{F}_n\left(z_i, k_i, \ell^I\right)\widetilde{\mathcal{F}_n}\left(z_i, -\bar{k}_i, -\ell^I\right)$, denoted chiral blocks. Chiral blocks have a universal monodromy behavior as the points are moved around one another or circled around the homology cycles of the surface, and these properties can be exploited (Of course, the monodromy of the chiral blocks plays a central role in calculations with the RNS formalism; see, e.g., [40], but in this review we will focus on the pure spinor formalism.) to propose pure spinor superstring integrands [41, 64]. More precisely, decomposing the chiral blocks into chiral kinematic correlators $\mathcal{K}\left(z_i, \ell^I\right)$ and a chiral Koba-Nielsen factor $\mathcal{I}_n$ (to be displayed below) as $\mathcal{F} = \langle \mathcal{K}_n \rangle \mathcal{I}_n$, the expression for the chiral correlator must be invariant under the combined homology shifts of vertex positions z_i and loop momenta ℓ^I around the A_I or B_I cycles:

以将 x^m 场的共形关联函数改写为对 ℓ_I 的积分形式。此时被积函数可分解为左动模与右动模的乘积，其概要形式为 $\mathcal{F}_n\left(z_i, k_i, \ell^I\right) \widetilde{\mathcal{F}_n}\left(z_i, -\bar{k}_i, -\ell^I\right)$，称为手征块。当顶点位置相互绕转或绕曲面同调环运动时，手征块具有普适的单值性，我们可以利用这些性质提出纯自旋 or 超弦被积函数 [41, 64](当然，手征块的单值性在 RNS 形式体系的计算中发挥核心作用，例如参见文献 [40]，但本综述我们将聚焦纯自旋 or 形式体系)。更准确地说，将手征块按 $\mathcal{F} = \langle\mathcal{K}_n\rangle \mathcal{I}_n$ 分解为手征运动学关联函数 $\mathcal{K}\left(z_i, \ell^I\right)$ 和手征 Koba-Nielsen 因子 $\mathcal{I}_n$ (后文将给出具体形式)，手征关联函数的表达式在顶点位置 z_i 和圈动量 ℓ^I 绕 A_I 或 B_I 环的组合同调平移下必须不变:

$$\mathcal{K}_n\left(z_i, k_i, \ell^I\right) = \mathcal{K}_n\left(z_i + \delta_{ij} A_J, k_i, \ell^I\right) \tag{40}$$

$$\mathcal{K}_n\left(z_i, k_i, \ell^I\right) = \mathcal{K}_n\left(z_i + \delta_{ij} B_J, k_i, \ell^I - 2\pi\delta_J^I k_j\right).$$

When viewed as a constraint on the chiral correlator, these invariances can be used as a guide to obtain superstring correlators [41,42,62-64].

当将这些不变性视为手征关联函数的约束时，它们可以作为得到超弦关联函数的指导 [41,42,62-64]。

Pure Spinor Superspace

纯旋量超空间

After all the nonzero modes of the worldsheet fields have been integrated out using OPEs, the correlator contains only the zero modes of conformal weight zero variables. In the minimal pure spinor formalism of [25], that means the zero modes of λ^α and θ^α , while in the non-minimal formalism, they can also include $\overline{\lambda}_\alpha$ and r_α variables. In the latter case, one can show that r_α can be converted to supersymmetric derivatives D_α while the pure spinors $\overline{\lambda}_\alpha$ can always be arranged to contract λ^α to produce scalar factors of $\left(\lambda\overline{\lambda}\right)$ which change the normalization factor. Therefore, the zero-mode integration (with a constant number of $\left(\lambda\overline{\lambda}\right)$ factors) can be done with the prescription [1]:

利用算子乘积展开对世界面场的所有非零模积分后，关联函数中仅保留共形权重为零变量的零模。在文献 [25] 的最小纯旋量形式中，这对应 λ^α 和 θ^α 的零模，而在非最小形式中，零模还可包含 $\overline{\lambda}_\alpha$ 和 r_α 变量。可以证明，在后一种情况下 r_α 可以转化为超对称导数 D_α，而纯旋量 $\overline{\lambda}_\alpha$ 总可以安排为收缩 λ^α，得到改变归一化因子的标量因子 $\left(\lambda\overline{\lambda}\right)$。因此，零模积分 (含固定数量的 $\left(\lambda\overline{\lambda}\right)$ 因子) 可通过以下规则 [1] 完成:

$$\left\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)\left(\theta\gamma_{mnp}\theta\right)\right\rangle = 2880. \tag{41}$$

This motivates the notion of pure spinor superspace [27], defined as expressions containing three pure spinors and an arbitrary number of SYM superfields composed of polarizations, momenta, and θ^α variables. The prescription (41) justifies the previous claim that terms of order $\theta^{>5}$ in (3) could be safely ignored. As a simple example of pure spinor superspace, one can consider extracting the supersymmetric expression of the massless open-string three-point amplitude at genus zero:

这引出了纯旋量超空间的概念 [27]，其定义为包含三个纯旋量和任意数量由极化、动量与 θ^α 变量构成的超杨-米尔斯超场的表达式。规则 (41) 证实了此前的结论:(3) 式中 $\theta^{>5}$ 阶项可以安全忽略。作为纯旋量超空间的一个简单例子，我们可以考虑提取零亏格零质量开弦三点振幅的超对称表达式:

$$\langle V_1 V_2 V_3\rangle = \Big[\frac{1}{64} k^2_m e^1_r e^2_n e^3_s \left\langle (\lambda\gamma^r\theta)(\lambda\gamma^s\theta)\left(\lambda\gamma_p\theta\right)(\theta\gamma^{pmn}\theta)\right\rangle$$

$$+\frac{1}{18} e^m_1 \left\langle (\lambda\gamma_m\theta)(\lambda\gamma_n\theta)\left(\lambda\gamma_p\theta\right)(\theta\gamma^n\chi_2)(\theta\gamma^p\chi_3)\right\rangle + \text{cyclic}\,(1,2,3)\Big]$$

$$= \frac{1}{2} e^m_1 f^{mn}_2 e^n_3 + e^1_m\,(\chi_2\gamma^m\chi_3) + \text{cyclic}\,(1,2,3)\,. \tag{42}$$

where we plugged in the θ expansions of (3) and kept only the terms with θ^5 . Moreover, we used:

其中我们代入了 (3) 式的 θ 展开，仅保留含 θ^5 的项。此外我们使用了:

$$\left\langle (\lambda\gamma_r\theta)(\lambda\gamma_s\theta)\left(\lambda\gamma_p\theta\right)(\theta\gamma^{pmn}\theta)\right\rangle = 64\delta^{mn}_{rs}, \tag{43}$$

$$\left\langle (\lambda\gamma^m\theta)(\lambda\gamma_n\theta)\left(\lambda\gamma_p\theta\right)(\theta\gamma^n\chi_2)(\theta\gamma^p\chi_3)\right\rangle = 18\,(\chi_2\gamma^m\chi_3)\,,$$

which can be derived from group-theory considerations (see appendix of [28]), momentum conservation $k^m_1 + k^m_2 + k^m_3 = 0$, and the transversality condition $(k_i\cdot e_i) = 0$.

它可以通过群论分析 (参见 [28] 附录)、动量守恒 $k^m_1 + k^m_2 + k^m_3 = 0$ 和横截性条件 $(k_i\cdot e_i) = 0$ 推导得出。

Multiparticle Superfields

多粒子超场

While four-point scattering amplitudes at one and two loops can be written down using the (single-particle) SYM superfields, the OPE contractions present at higher points lead to linear combinations of SYM superfields whose patterns are captured by so-called multiparticle superfields, describing multiple string states at the same time. Not only they encode the numerators associated with OPE singularities, but they are also designed in a way that removes BRST-exact pieces and total derivatives. The end result displays covariant BRST transformations and generalized Jacobi identities [31] - the latter property is particularly useful for describing Bern-Carrasco-Johansson color/kinematics duality [29].

虽然单圈和双圈四点散射振幅可以用 (单粒子)SYM 超场写出，高阶点存在的 OPE 缩并会得到 SYM 超场的线性组合，这些组合的结构可由所谓的多粒子超场描述，它同时描述多个弦态。它不仅编码了与 OPE 奇点相关的分子，还被设计用来移除 BRST 恰当项和全导数。最终结果呈现出协变 BRST 变换和推广的雅可比恒等式 [31]——后者性质对描述 Bern-Carrasco-Johansson 色/运动学对偶 [29] 尤其有用。

The two-particle superfields generalizing the standard superfields of (2) are given by [66]

推广 (2) 中标准超场的双粒子超场由文献 [66] 给出

$$A_\alpha^{12} = \frac{1}{2}\left[A_\alpha^2 (k_2 \cdot A_1) + A_2^m(\gamma_m W_1)_\alpha - (1 \leftrightarrow 2)\right],$$

$$A_{12}^m = \frac{1}{2}\left[A_2^m (k_2 \cdot A_1) + A_p^1 F_2^{pm} + (W_1\gamma^m W_2) - (1 \leftrightarrow 2)\right],$$

$$W_{12}^\alpha = \frac{1}{4}(\gamma_{mn}W_2)^\alpha F_1^{mn} + W_2^\alpha (k_2 \cdot A_1) - (1 \leftrightarrow 2), \tag{44}$$

$$F_{12}^{mn} = F_2^{mn} (k_2 \cdot A_1) + \frac{1}{2}F_2^{[m}{}_p F_1^{n]p} + k_1^{[m}\left(W_1\gamma^{n]}W_2\right) - (1 \leftrightarrow 2),$$

and satisfy

且满足

$$D_\alpha A_\beta^{12} + D_\beta A_\alpha^{12} = \gamma_{\alpha\beta}^m A_m^{12} + (k_1 \cdot k_2)\left(A_\alpha^1 A_\beta^2 + A_\beta^1 A_\alpha^2\right),$$

$$D_\alpha A_{12}^m = \gamma_{\alpha\beta}^m W_{12}^\beta + k_{12}^m A_\alpha^{12} + (k_1 \cdot k_2)\left(A_\alpha^1 A_2^m - A_\alpha^2 A_1^m\right),$$

$$D_\alpha W_{12}^\beta = \frac{1}{4}(\gamma_{mn})_\alpha{}^\beta F_{12}^{mn} + (k_1 \cdot k_2)\left(A_\alpha^1 W_2^\beta - A_\alpha^2 W_1^\beta\right),$$

$$D_\alpha F_{12}^{mn} = k_{12}^{[m}\left(\gamma^{n]}W_{12}\right)_\alpha + (k_1 \cdot k_2)\Big[A_\alpha^1 F_2^{mn} + A_1^{[n}\left(\gamma^{m]}W_2\right)_\alpha$$

$$- (1 \leftrightarrow 2)\Big] \, .$$

(45)

These equations of motion have the same form as in the single-particle case (2) with additional corrections proportional to $\left(k^1 \cdot k^2\right)$. The construction of (local) multi-particle superfields of arbitrary multiplicity leads to superfields labeled by words $P = p_1p_2p_3\ldots$ or by arbitrary nested commutators $P = [\ldots[[p_1, p_2], p_3], \ldots]$ (e.g., A_{1234}^m or $F_{[1,[2,3]]}^{mn}$) and can be found in [32, 66] .

这些运动方程与 (2) 中单粒子情况形式相同，仅带有正比于 $\left(k^1 \cdot k^2\right)$ 的额外修正项。任意重数的 (定域) 多粒子超场的构造得到了由字 $P = p_1p_2p_3\ldots$ 或任意嵌套对易子 $P = [\ldots[[p_1, p_2], p_3], \ldots]$ 标记的超场 (例如 A_{1234}^m 或 $F_{[1,[2,3]]}^{mn}$)，相关构造可参见 [32, 66] 。

Superstring Amplitudes with Pure Spinors

纯旋子超弦振幅

The pure spinor prescription to compute genus- g amplitudes relies on the basic fact that the OPE analysis of primary operators determines a meromorphic function of the vertex positions due to its poles and residues. In the absence of monodromy such as at genus zero, this completely determines the correlator, but this is no longer true at higher genus. On a surface of higher genus, the existence of holomorphic one-forms implies that the knowledge of the positions and residues of the poles from the OPE analysis no longer suffices to completely determine the correlator; the regular terms contain nontrivial information. In principle, the zero modes provide the additional information to find the complete correlator [74]. However, sometimes this is impractical to follow systematically and the calculation benefits from the practical requirements of homology and BRST invariance (It is worth mentioning that several amplitudes computed in this manner used the "minimal" pure spinor formalism and its simpler pure spinor superspace expressions depending only on the zero modes of λ^α and θ^α (the expressions in the non-minimal formalism also depend on $\overline{\lambda}_\alpha, r_\alpha$ in intermediate stages).) constraints to be discussed below.

计算亏格- g 振幅的纯旋子方案基于一个基本事实: 初算子的 OPE 分析会通过极点与残差确定顶点位置的亚纯函数。在不存在单值性的情况 (例如零亏格情形)，这可以完全确定关联函数，但在高亏格情形下不再成立。在高亏格曲面上，全纯一形式的存在意味着，仅通过 OPE 分析得到的极点位置与残差，已经不足以完全确定关联函数；正则项包含非平凡信息。原则上，零模可提供确定完整关联函数所需的额外信息 [74]。但这种系统推导的方法有时并不实用，利用同调与 BRST 不变性的实际约束 (值得一提的是，按该方法计算的若干振幅使用了“极小”纯旋子形式论，其纯旋子超空间表达式更简单，仅依赖 λ^α 和 θ^α 的零模；非极小形式论的表达式在中间阶段也会依赖 $\overline{\lambda}_\alpha, r_\alpha$) 开展计算会更方便，我们将在下文中讨论这些约束。

Homology invariance The introduction of loop momentum integrals with the chiral-splitting formalism had to pass the consistency check that the integrated amplitudes were single-valued as a function $\widehat{f}(z_i)$ of the vertex positions z_i after the loop momentum was integrated out [39]. However, a stronger constraint was proposed (It was initially dubbed "monodromy invariance," and it led to the development of generalized elliptic integrands (GEI) in the context of genus-one string amplitudes [63].) in [63, 67]: that the chiral integrands, viewed as a function $f(\ell_I, z_i)$ of both the loop momenta ℓ_I and vertex positions z_i , should be strictly single-valued under the monodromies of the loop momentum and the vertex positions as they move around A_I and B_I cycles:

同调不变性引入手征分裂形式论的圈动量积分后，必须通过一致性检验: 积分掉圈动量后，积分振幅作为顶点位置 z_i 的函数 $\widehat{f}(z_i)$ 是单值的 [39]。然而文献 [63, 67] 提出了一个更强的约束 (它最初被称为“单环不变性”，该概念推动了亏格一弦振幅背景下广义椭圆被积函数 (GEI) 的发展 [63]): 将手征被积函数看作圈动量 ℓ_I 和顶点位置 z_i 的函数 $f(\ell_I, z_i)$ ，当圈动量和顶点位置绕 A_I 和 B_I 闭链运动时，手征被积函数在单变换下应当严格单值:

$$f\left(z_i', \ell_I'\right) = f\left(z_i, \ell_I\right), \begin{cases} A_I - \text{cycle}: & \left(z_i', \ell_I'\right) = \left(z_i + \delta_{ij} A_J, \ell_I\right) \\ B_I - \text{cycle}: & \left(z_i', \ell_I'\right) = \left(z_i + \delta_{ij} B_J, \ell_I - 2\pi i \delta_I^J k_j\right) \end{cases}$$

(46)

That is, the chiral integrands should be single-valued before the loop momentum is integrated out. This requirement interlocks the different sectors of the integrands with different powers of loop momenta with predictive consequences: it can be used to constrain and obtain the superstring integrands themselves.

也就是说，手征被积函数在积分掉圈动量之前就应当是单值的。该要求将不同圈动量幂次的被积函数 sectors 相互关联，产生了可预测的结论: 它可用于约束并得到超弦被积函数本身。

This requirement of homology invariance was used in [62-64] to determine the integrands of the five-, six-, and seven-point massless amplitudes at genus one, and in [41] to obtain the massless five-point integrand at genus two.

> 这种同调不变性要求被文献 [62-64] 用于确定亏格一的 5 点、6 点和 7 点无质量振幅的被积函数，被文献 [41] 用于得到亏格二的无质量 5 点被积函数。

BRST invariance Superstring scattering amplitudes must be spacetime super-symmetric and gauge invariant. As explained in detail in [1], the cohomology prescription (41) to integrate out the pure spinor zero modes leads to gauge-invariant and supersymmetric expressions if the pure spinor superspace expression is BRST invariant. Recall that when the BRST charge (11) acts on superfield expressions containing only x^m, θ^α (and possibly λ^α), the OPE (30) implies

> BRST 不变性超弦散射振幅必须满足时空超对称与规范不变性。正如文献 [1] 的详细说明，如果纯旋子超空间表达式是 BRST 不变的，那么积分掉纯旋子零模的上同调方案 (41) 就会给出规范不变且超对称的表达式。回顾一下，当 BRST 电荷 (11) 作用在仅包含 x^m, θ^α (可能还包含 λ^α) 的超场表达式上时，OPE(30) 给出

$$QK(x,\theta) = \lambda^\alpha D_\alpha K(x,\theta) \tag{47}$$

where D_α is the superspace derivative (1) and λ^α is the pure spinor. As we will see, this equation plays an important role in the study of the BRST cohomology properties of string scattering amplitudes. More precisely, if the outcome of the OPEs among the vertices is written in pure spinor superspace as $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(e_i, k_i, \xi_i, \theta) \rangle$ where e_i, ξ_i and k_i represent a collection of bosonic and fermionic polarizations and their momenta, then the amplitude prescription will give rise to gauge- invariant and supersymmetric expressions if

> 其中 D_α 是超空间导数 (1)，λ^α 是纯旋子。我们将会看到，该方程在研究弦散射振幅的 BRST 上同调性质中发挥重要作用。更准确地说，如果顶点间 OPE 的结果可以在纯旋子超空间中写为 $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(e_i, k_i, \xi_i, \theta) \rangle$ ，其中 e_i, ξ_i 和 k_i 代表一组玻色极化、费米极化及其动量，那么只要满足条件，振幅方案就会给出规范不变且超对称的表达式:

$$Q\left(\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(e_i, k_i, \xi_i, \theta)\right) = 0, \tag{48}$$

$$\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(e_i, k_i, \xi_i, \theta) \neq Q\Omega.$$

This implies that the superspace expressions of arbitrary scattering amplitudes must be in the cohomology of the BRST charge. This requirement together with the OPE structure of the genus-zero pure spinor prescription is enough to completely determine the tree-level scattering amplitudes of ten-dimensional SYM theory [59,60].

> 这意味着任意散射振幅的超空间表达式必须处于 BRST 荷的上同调中。该要求结合零亏格纯自旋 or 方案的 OPE 结构，足以完全确定十维超对称杨-米尔斯理论的树级散射振幅 [59,60]。

Genus Zero

亏格零

SYM tree amplitudes The knowledge that the genus-zero superstring amplitudes reduce to ten-dimensional SYM tree amplitudes [81] has a powerful consequence: the tree amplitudes in field theory have the same superfield structure as their string theory counterparts. This led to the suggestion that the BRST cohomology structure of pure spinor superspace expressions inspired by the pure spinor prescription could be used to completely fix the form of the SYM tree amplitudes [59]. Using multiparticle superfields, the first nonvanishing tree amplitudes were found to be

SYM 树振幅亏格零超弦振幅约化为十维 SYM 树振幅这一结论 [81] 有一个有力推论: 场论中的树振幅与弦论对应对象具有相同的超场结构。这启发了如下猜想: 由纯旋子方案启发得到的纯旋子超空间表达式的 BRST 上同调结构，可以完全确定 SYM 树振幅的形式 [59]。利用多粒子超场，首个非零树振幅被发现为

$$A(1,2,3) = \langle V_1 V_2 V_3 \rangle, \tag{49}$$

$$A(1,2,3,4) = \frac{\langle V_{12} V_3 V_4 \rangle}{s_{12}} + \frac{\langle V_1 V_{23} V_4 \rangle}{s_{23}},$$

$$A(1,2,3,4,5) = \frac{\langle V_{123} V_4 V_5 \rangle}{s_{12} s_{45}} + \frac{\langle V_{321} V_4 V_5 \rangle}{s_{23} s_{45}} + \frac{\langle V_{12} V_{34} V_5 \rangle}{s_{12} s_{34}} + \frac{\langle V_1 V_{234} V_5 \rangle}{s_{23} s_{51}} + \frac{\langle V_1 V_{432} V_5 \rangle}{s_{34} s_{51}}.$$

The regular structure of the BRST variation of certain nonlocal multiparticle superfield building blocks M_P , the Berends-Giele currents, and various other hints led to the general n -point expression for SYM tree amplitudes in [60]:

某些非局域多粒子超场构造块 M_P (即贝伦兹-吉勒流) 的 BRST 变分具有正则结构，结合诸多其他线索，文献 [60] 给出了 SYM 树振幅的一般 n 点表达式:

$$A(P,n) = \sum_{XY=P} \langle M_X M_Y M_n \rangle \tag{50}$$

where $XY = P$ represents the sum over all deconcatenations of the word P into the words X and Y (including the empty word provided we define $M_\varnothing := 0$). In this language, the amplitudes (49) become:

其中 $XY = P$ 代表将字 P 拆分为字 X 和 Y 的所有解连接求和 (若我们定义 $M_\varnothing := 0$ ，则包含空字的情况)。在此表述下，振幅 (49) 可写为:

$$A(1,2,3) = \langle M_1 M_2 M_3 \rangle, \tag{51}$$

$$A(1,2,3,4) = \langle M_{12} M_3 M_4 \rangle + \langle M_1 M_{23} M_4 \rangle,$$

$$A(1,2,3,4,5) = \langle M_{123} M_4 M_5 \rangle + \langle M_{12} M_{34} M_5 \rangle + \langle M_1 M_{234} M_5 \rangle .$$

The explicit expressions for the Berends-Giele currents in terms of multiparticle superfields, the first of which are given by

贝伦兹-吉勒流可以用多粒子超场表示出显式表达式，其中前几个为

$$M_1 = V_1, \; M_{12} = \frac{V_{12}}{s_{12}}, \; M_{123} = \frac{V_{123}}{s_{12} s_{123}} + \frac{V_{321}}{s_{23} s_{123}}, \tag{52}$$

can be constructed in a multitude of ways (see [82]). Their BRST variation admits a simple all-order form:

可通过多种方法构造 (参见 [82])。它们的 BRST 变分具有简洁的全阶形式:

$$Q M_P = \sum_{XY=P} M_X M_Y \tag{53}$$

from which it easily follows that the superfield expression in (50) is BRST closed. It is also not BRST-exact, and therefore it is in the cohomology of the BRST charge. To see this, note that M_P contains a divergent propagator $1/s_P$ in the phase space of $|P| + 1 = n$ massless particles, so one cannot write the superfields in (50) as $Q(M_P M_n)$. In other words, $M_P M_n$ is not an allowable BRST ancestor, which explains why $\left\langle \sum_{XY=P} M_X M_Y M_n \right\rangle \neq 0$.

由此很容易得知，(50) 中的超场表达式是 BRST 闭的。它同时不是 BRST 恰当的，因此属于 BRST 荷的上同调。要理解这一点，注意到 M_P 在 $|P| + 1 = n$ 个无质量粒子的相空间中包含发散传播子 $1/s_P$ ，因此我们无法将 (50) 中的超场写为 $Q(M_P M_n)$ 。换句话说，$M_P M_n$ 不是合格的 BRST 原初项，这解释了为什么 $\left\langle \sum_{XY=P} M_X M_Y M_n \right\rangle \neq 0$ 。

The n -point superstring disk correlator The general n -point disk correlator of massless string states was computed in [61] using multiparticle superfield techniques to capture the OPE singularities of vertex operators. The result can be written as a sum over $(n-3)!$ SYM field-theory tree amplitudes (50) as follows:

n 点超弦圆盘关联函数文献 [61] 利用多粒子超场技术处理顶点算符的 OPE 奇点，计算了无质量弦态的一般 n 点圆盘关联函数。结果可以写为对 $(n-3)!$ 个 SYM 场论树振幅 (50) 的求和，形式如下:

$$\mathcal{A}_n(P) = (2\alpha')^{n-3} \int d\mu_P^n \left[\prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} A(1,2,\ldots,n) + \mathrm{perm}(2,3,\ldots,n-2) \right], \tag{54}$$

where $\int d\mu_P^n$ is a shorthand for the integration over the vertex positions with integration domain $D(P)$ and weighted by the genus-zero Koba-Nielsen factor $\int_{D(P)} \prod_{j=2}^{n-2} dz_j \prod_{1 \leq i < j}^{n-1} |z_{ij}|^{-2\alpha' s_{ij}}$. This result motivated the development of a method [35] to obtain the α' expansion of the integrals in (54). In addition, the conclusion that there is a $(n-3)!$ basis of tree amplitudes in the work of Bern, Carrasco, and Johansson [29] becomes manifest as the left-hand side must reduce, in the limit as $\alpha' \to 0$, to a color-ordered SYM tree amplitude $A(P)$

with arbitrary ordering P , which in turn is expanded in terms of $(n-3)!$ tree amplitudes on the right-hand side. For an in-depth discussion of these matters, see [82].

其中 $\int d\mu_P^n$ 是顶点位置积分的简写，该积分的积分域为 $D(P)$ ，权重为零格矢神保-尼尔森因子 $\int_{D(P)} \prod_{j=2}^{n-2} dz_j \prod_{1\leq i<j}^{n-1} |z_{ij}|^{-2\alpha' s_{ij}}$ 。这一结果推动了方法 [35] 的发展，用于得到式 (54) 中积分的 α' 展开。此外，Bern、Carrasco 与 Johansson 在文献 [29] 中提出的树图振幅存在 $(n-3)!$ 组基的结论在此变得十分清晰: 当取 $\alpha' \to 0$ 极限时，左侧必须约化为任意序 P 的色序 SYM 树图振幅 $A(P)$ ，而后者又可在右侧用 $(n-3)!$ 个树图振幅展开。有关这些问题的深入讨论参见文献 [82]。

Genus One

亏格一

We are now going to showcase some of the results obtained with the pure spinor formalism at genus one. For the open string, the amplitudes have the general form:

我们现在将展示纯自旋形式下亏格一的部分结果。对于开弦，振幅的一般形式如下:

$$\mathcal{A}_n = \sum_{\text{top}} C_{\text{top}} \int_{D_{\text{top}}} d\tau dz_2 dz_3 \dots dz_n \int d^D\ell \, |\mathcal{I}_n(\ell)| \langle \mathcal{K}_n(\ell) \rangle , \tag{55}$$

with $\langle \dots \rangle$ denoting the zero-mode integration prescription (32), which will be presented in the examples below as pure spinor superspace expressions in terms of zero modes of λ^α and θ^α . The integration domains D_{top} for the modular parameter τ and vertex positions z_j must be chosen according to the topologies of a cylinder or a Möbius strip with associated color factors C_{top} . The integration over loop momenta ℓ must be performed as a consequence of the chiral-splitting method, which, in turn, allows to derive massless closed-string one-loop amplitudes from an integrand of double-copy form:

其中 $\langle \dots \rangle$ 表示零模积分规则 (32)，下文示例会将其表示为基于 λ^α 和 θ^α 零模的纯自旋超空间表达式。必须根据圆柱或莫比乌斯带的拓扑选择模参数 τ 和顶点位置 z_j 的积分域 D_{top} ，并对应关联颜色因子 C_{top} 。手征分裂方法要求必须对圈动量 ℓ 做积分，该方法进而可以从双拷贝形式的被积函数导出无质量闭弦单圈振幅:

$$\mathcal{M}_n = \int_{\mathcal{F}} d^2\tau d^2 z_2 d^2 z_3 \dots d^2 z_n \int d^D\ell |\mathcal{I}_n(\ell)|^2 \langle \mathcal{K}_n(\ell) \rangle \langle \tilde{\mathcal{K}}_n(-\ell) \rangle , \tag{56}$$

with $\mathcal{F}$ denoting the fundamental domain for inequivalent tori with respect to the modular group. Both expressions (55) and (56) involve the universal one-loop Koba-Nielsen factor:

其中 $\mathcal{F}$ 表示模群下不等价环面的基本域。式 (55) 和 (56) 都包含通用单圈小林-尼尔森因子:

$$\mathcal{I}_n(\ell) \equiv \exp\left(\sum_{i<j}^n s_{ij} \log \theta_1(z_{ij}, \tau) + \sum_{j=1}^n z_j (\ell \cdot k_j) + \frac{\tau}{4\pi i} \ell^2 \right), \tag{57}$$

with light-like external momenta k_j and $s_{ij} \equiv k_i \cdot k_j$ as well as $z_{ij} \equiv z_i - z_j$.

其中类光外动量为 k_j 、$s_{ij} \equiv k_i \cdot k_j$ 以及 $z_{ij} \equiv z_i - z_j$ 。

The Eisenstein-Kronecker series As pointed out above, knowing the singularity structure of the superstring correlators is not enough to reconstruct the full meromorphic integrand as a function of the vertex positions, as crucial information from the non-singular parts is needed. In [34], a generating series of worldsheet functions was proposed that contained an infinite tower of functions $g^{(n)}(z)$ for $n \geq 0$ on a complex elliptic curve describing a genus-one surface with modulus τ . These functions turn out to have the correct properties to capture both the singular part of superstring correlators with $g^{(1)}$ and the non-singular pieces with $g^{(n)}, n \geq 2$. More precisely, these functions are constructed via the Laurent series of the Eisenstein-Kronecker series $F(z, \alpha, \tau)$ [36]:

艾森斯坦-克罗内克级数正如上文指出，仅了解超弦关联函数的奇点结构，不足以重构作为顶点位置函数的完整亚纯被积函数——我们还需要非奇异部分的关键信息。文献 [34] 提出了一个世界面函数的生成级数，它包含描述亏格一曲面 (模为 τ) 的复椭圆曲线上 $n \geq 0$ 对应函数 $g^{(n)}(z)$ 的无穷层塔。这些函数具备合适的性质，可以同时捕捉含 $g^{(1)}$ 的超弦关联函数的奇异部分和含 $g^{(n)}, n \geq 2$ 的非奇异部分。更准确地说，这些函数通过艾森斯坦-克罗内克级数 $F(z, \alpha, \tau)$ [36] 的洛朗级数构造:

$$F(z, \alpha, \tau) \equiv \frac{\theta_1'(0, \tau)\, \theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\, \theta_1(\alpha, \tau)} = \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z, \tau) \tag{58}$$

where $\theta_1(z, \tau)$ is the odd Jacobi theta function $\left(q = e^{2\pi i \tau}\right)$

其中 $\theta_1(z, \tau)$ 是奇雅可比 θ 函数 $\left(q = e^{2\pi i \tau}\right)$

$$\theta_1(z, \tau) \equiv 2 i q^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} \left(1 - q^j\right) \prod_{j=1}^{\infty} \left(1 - e^{2\pi i z} q^j\right) \prod_{j=1}^{\infty} \left(1 - e^{-2\pi i z} q^j\right), \tag{59}$$

satisfying $\theta_1(z + 1, \tau) = -\theta_1(z, \tau)$ and $\theta_1(z + \tau, \tau) = -e^{-\pi i \tau} e^{-2\pi i z} \theta_1(z, \tau)$ as z is moved around the A or B cycle. In addition, $\theta_1'(z, \tau) = \partial_z \theta_1(z, \tau)$. The functions $g^{(n)}$ for the first few cases are $g^{(0)}(z, \tau) = 1$:

当 z 沿 A 或 B 周期平移时满足 $\theta_1(z + 1, \tau) = -\theta_1(z, \tau)$ 和 $\theta_1(z + \tau, \tau) = -e^{-\pi i \tau} e^{-2\pi i z} \theta_1(z, \tau)$ ，此外还有 $\theta_1'(z, \tau) = \partial_z \theta_1(z, \tau)$ 。前几种情况的函数 $g^{(n)}$ 为 $g^{(0)}(z, \tau) = 1$:

$$g^{(1)}(z, \tau) = \partial \log \theta_1(z, \tau), \; g^{(2)}(z, \tau) = \frac{1}{2}\left[\left(\partial \log \theta_1(z, \tau)\right)^2 - \wp(z, \tau)\right], \tag{60}$$

where $\wp(z, \tau) = -\partial^2 \log \theta_1(z, \tau) - G_2(\tau)$ is the Weierstrass function and $G_{2k}(\tau)$ are holomorphic Eisenstein series.

其中 $\wp(z, \tau) = -\partial^2 \log \theta_1(z, \tau) - G_2(\tau)$ 是魏尔斯特拉斯函数，$G_{2k}(\tau)$ 是全纯艾森斯坦级数。

The function $g^{(1)}(z, \tau)$ is singular as $z \to 0$ while all $g^{(n)}(z, \tau)$ with $n \geq 2$ are non-singular in this limit. In addition, all $g^{(n)}(z, \tau)$ are single-valued around

函数 $g^{(1)}(z, \tau)$ 在此极限下表现为 $z \to 0$ 型奇异，而所有满足 $n \geq 2$ 的 $g^{(n)}(z, \tau)$ 在此极限下非奇异。此外，所有 $g^{(n)}(z, \tau)$ 在

the A -cycle as $z \to z+1$ but have nontrivial monodromy around the B -cycle as $z \to z+\tau$

A 圈周围是单值的，形式为 $z \to z+1$ ，但在 B 圈周围具有非平凡单值性，形式为 $z \to z+\tau$

$$
g^{(n)}(z+\tau,\tau)=\sum_{k=0}^{n}\frac{(-2\pi i)^k}{k!}g^{(n-k)}(z,\tau). \tag{61}
$$

For instance, $g^{(1)}(z+\tau,\tau)=-2\pi i$ and $g^{(2)}(z+\tau,\tau)=-2\pi i g^{(1)}(z,\tau)+\frac{1}{2}(2\pi i)^2$. The singularity structure of these functions as well as their monodromies in a genus-one surface provided valuable information to constrain and obtain [62-64] the genus-one n -point superstring correlators for $n \le 7$ using the homology invariance principle discussed above. The shorthand $g_{ij}^{(n)} := g^{(n)}(z_i-z_j,\tau)$ will be used below, and it will be convenient to define a linearized B -cycle monodromy operator D :

例如 $g^{(1)}(z+\tau,\tau)=-2\pi i$ 和 $g^{(2)}(z+\tau,\tau)=-2\pi i g^{(1)}(z,\tau)+\frac{1}{2}(2\pi i)^2$ 。这些函数的奇异结构及其在亏格一表面上的单值性，为利用上文讨论的同调不变性原理约束并得到 $n \le 7$ 情况下亏格一 n 点超弦关联函数 [62-64] 提供了重要信息。下文将使用简写 $g_{ij}^{(n)} := g^{(n)}(z_i-z_j,\tau)$ ，且方便定义线性化 B 圈单值算符 D :

$$
D=-\frac{1}{2\pi i}\sum_{j=1}^{n}\Omega_j\delta_j \tag{62}
$$

where Ω_j are formal variables that capture the B -cycle monodromies around z_j generated by the formal operator δ_j with action $\delta_j\ell=-2\pi i k_j$ and $\delta_j g_{jm}^{(n)}=-2\pi i g_{jm}^{(n-1)}$ for $n \ge 1$ as well as $\delta_j g_{jm}^{(0)}=0$ and $\delta_j g_{im}^{(n)}=0$ for all $i,m \neq j$. As discussed in [63], there is a remarkable duality relating the operator D with the BRST charge Q .

其中 Ω_j 是形式变量，用于刻画形式算符 δ_j 诱导的绕 z_j 的 B 圈单值性，该算符对 $n \ge 1$ 的作用为 $\delta_j\ell=-2\pi i k_j$ 和 $\delta_j g_{jm}^{(n)}=-2\pi i g_{jm}^{(n-1)}$ ，对所有 $i,m \neq j$ 的作用为 $\delta_j g_{jm}^{(0)}=0$ 和 $\delta_j g_{im}^{(n)}=0$ 。正如文献 [63] 所述，存在一个显著的对偶性将算符 D 与 BRST 荷 Q 联系起来。

BRST building blocks The other ingredient used to obtain the genus-one super-string correlators was the BRST invariance property of the integrands. This was addressed by the construction of BRST building blocks with covariant BRST transformations, using multiparticle superfield techniques in combination with pure spinor zero-mode analysis and group theory to constrain the appearance of superfields. This led to the definition of multiple BRST building blocks with different BRST transformation properties allowing for the construction of BRST invariants in the pure spinor cohomology.

BRST 结构基元得到亏格一超弦关联函数的另一项基础是被积函数的 BRST 不变性。这一问题通过构造具有协变 BRST 变换的 BRST 结构基元解决: 结合多粒子超场技术、纯自旋零模分析与群论对超场的出现方式加以约束，最终得到了具有不同 BRST 变换性质的多个 BRST 结构基元，可用于在纯自旋上同调中构造 BRST 不变量。

For instance, the zero-mode sector with four d_α zero modes from the b ghost suggests the scalar building blocks:

例如，来自 b 鬼的四个 d_α 零模的零模扇区给出如下标量结构基元:

$$T_{A,B,C} = \frac{1}{3}(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F_C^{mn} + \text{cyclic}(A,B,C)\,. \tag{63}$$

in terms of multiparticle superfields labeled by words A, B, C . Their BRST variation following (47) is given by $\big(k_\varnothing \equiv 0\big)$:

> 它们可以用由字母 A, B, C 标记的多粒子超场表示。根据式 (47)，其 BRST 变分为 $\big(k_\varnothing \equiv 0\big)$:

$$QT_{A,B,C} = \sum_{\substack{A=XjY\\ Y=R\shuffle S}} (k_X \cdot k_j)\big[V_{XR}T_{jS,B,C} - V_{jR}T_{XS,B,C}\big] + (A \leftrightarrow B, C)\,, \tag{64}$$

where $\shuffle$ denotes the shuffle product defined iteratively by [73]

> 其中 $\shuffle$ 表示由文献 [73] 迭代定义的洗牌积，即

$$\varnothing \shuffle P = P \sqcup \varnothing \;:=\; P,\; iP \sqcup jQ \;:=\; i(P \sqcup jQ) + j(Q \sqcup iP)\,, \tag{65}$$

for letters i and j , words P and Q with $\varnothing$ representing the empty word. For example, $1 \sqcup 23 = 123 + 213 + 231$.

> 对字母 i 和 j 、字 P 和 Q 成立，其中 $\varnothing$ 表示空字。例如 $1 \sqcup 23 = 123 + 213 + 231$ 。

For an illustration of (64), the BRST variations of all $T_{A,B,C}$ up to multiplicity five are given by

> 为说明式 (64)，我们给出所有 multiplicity 不超过 5 的 $T_{A,B,C}$ 的 BRST 变分如下:

$$QT_{1,2,3} = 0 \tag{66}$$

$$QT_{12,3,4} = (k_1 \cdot k_2)\big[V_1 T_{2,3,4} - V_2 T_{1,3,4}\big]\,,$$

$$QT_{123,4,5} = (k_1 \cdot k_2)\big[V_1 T_{23,4,5} + V_{13} T_{2,4,5} - V_2 T_{13,4,5} - V_{23} T_{1,4,5}\big]$$

$$+ (k_{12} \cdot k_3)\big[V_{12} T_{3,4,5} - V_3 T_{12,4,5}\big]\,,$$

$$QT_{12,34,5} = (k_1 \cdot k_2)\big[V_1 T_{2,34,5} - V_2 T_{1,34,5}\big] + (12 \leftrightarrow 34)\,.$$

Other zero-mode contributions from the b ghost give rise to tensorial building blocks with an arbitrary number of vector indices. For simplicity, the vector building block has the form:

> b 鬼的其他零模贡献会生成带有任意多个矢量指标的张量结构基元。为简化起见，矢量结构基元的形式为:

$$T^m_{A,B,C,D} \equiv \big[A^m_A T_{B,C,D} + (A \leftrightarrow B, C, D)\big] + W^m_{A,B,C,D} \tag{67}$$

with

其中

$$W^m_{A,B,C,D} = \frac{1}{12}(\lambda\gamma_n W_A)(\lambda\gamma_p W_B)(W_C\gamma^{mnp}W_D) + (A,B \mid A,B,C,D) \tag{68}$$

with the notation $(A_1,\dots,A_p \mid A_1,\dots,A_n)$ instructing to sum over all possible ways to choose p elements $A_1,A_2,\dots,A_p$ out of the set $\{A_1,\dots,A_n\}$, for a total of $\begin{pmatrix} n \\ p \end{pmatrix}$ terms.

其中记号 $(A_1,\dots,A_p \mid A_1,\dots,A_n)$ 表示对从集合 $\{A_1,\dots,A_n\}$ 中选出 p 个元素 $A_1,A_2,\dots,A_p$ 的所有可能方式求和，总共有 $\begin{pmatrix} n \\ p \end{pmatrix}$ 项。

The BRST transformation of (67) is given by

(67) 的 BRST 变换由下式给出

$$\begin{aligned} QT^m_{A,B,C,D} = k^m_A V_A T_{B,C,D} + \sum_{\substack{A=XjY \\ Y=R\sqcup S}} (k_X \cdot k_j)\left[V_{XR}T^m_{jS,B,C,D} - V_{jR}T^m_{XS,B,C,D}\right] \\ + (A \leftrightarrow B,C,D), \end{aligned} \tag{69}$$

for example,

例如,

$$QT^m_{1,2,3,4} = k^m_1 V_1 T_{2,3,4} + (1 \leftrightarrow 2,3,4), \tag{70}$$

$$QT^m_{12,3,4,5} = \left[k^m_{12}V_{12}T_{3,4,5} + (12 \leftrightarrow 3,4,5)\right] + (k_1 \cdot k_2)\left(V_1 T^m_{2,3,4,5} - V_2 T^m_{1,3,4,5}\right).$$

Other building blocks were defined in [62] to capture the gauge anomaly of the field-theory SYM integrands that disappear in the $SO(32)$ superstring [49,50].

其他构造块已在文献 [62] 中定义，用来描述场论超对称杨-米尔斯被积函中会在 $SO(32)$ 超弦中消失的规范反常 [49,50]。

Four points At genus one, the simplest scattering amplitude with four massless states computed in 1982 by Green and Schwarz [58] was reproduced in a 2004 calculation using the minimal pure spinor formalism [25]. A salient feature of this calculation is the absence of OPE singularities among the vertices; the amplitude is completely determined by the pure spinor zero modes. The result of the correlator in the conventions of (55) is given by

四点亏格一情况下，1982 年格林与施瓦茨计算得到的最简四质量态散射振幅 [58]，在 2004 年用最小纯自旋形式框架 [25] 中得到了重现。该计算的一个突出特点是顶点之间不存在 OPE 奇点；振幅完全由纯自旋零模确定。相关子在 (55) 的约定下的结果由下式给出

$$\mathcal{K}_4(\ell) = V_1 T_{2,3,4}, \tag{71}$$

and its zero-mode evaluation can be written in terms of the tree-level SYM amplitude A^{SYM} as follows

其零模计算可以用树图级超对称杨-米尔斯振幅 A^{SYM} 表示如下

$$\langle V_1 T_{2,3,4} \rangle = s_{12} s_{23} A^{\mathrm{SYM}}(1,2,3,4). \tag{72}$$

For Neveu-Schwarz external states, the zero-mode evaluation of (72) yields the famous t_8 tensor, $\langle V_1 T_{2,3,4} \rangle = \frac{1}{2} t_8(f_1, f_2, f_3, f_4)$ where

对于内态为纳维-施瓦茨的情况，(72) 的零模计算给出了著名的 t_8 张量，$\langle V_1 T_{2,3,4} \rangle = \frac{1}{2} t_8(f_1, f_2, f_3, f_4)$ 其中

$$t_8(f_1, f_2, f_3, f_4) = \mathrm{tr}(f_1 f_2 f_3 f_4) - \frac{1}{4} \mathrm{tr}(f_1 f_2)\, \mathrm{tr}(f_3 f_4) + \mathrm{cyclic}(2,3,4), \tag{73}$$

and tr represents a trace over Lorentz indices, for example, $\mathrm{tr}(f_1, f_2) = f_1^{mn} f_2^{nm}$.

tr 表示对洛伦兹指标求迹，例如 $\mathrm{tr}(f_1, f_2) = f_1^{mn} f_2^{nm}$ 。

Five points At five points, an analysis of the structure of the superstring correlator arising from the pure spinor prescription (26) reveals that it is composed of two sectors: one containing a loop momentum contracting a vectorial combination of superfields and no OPE singularities and another with no loop momentum and with singularities as vertex positions collide multiplying a collection of superfields with no free vector indices. Combining this information with the BRST transformation properties of scalar (64) and vectorial building blocks (69) as well as the monodromy properties of the functions $g^{(n)}(z,\tau)$ and ℓ^m yields the proposal for the five-point correlator:

五点五点情况下，对纯自旋规则 (26) 得到的超弦相关子结构分析表明，它由两个部分构成: 一个部分包含圈动量收缩超场的矢量组合，不存在 OPE 奇点；另一部分不含圈动量，当顶点位置碰撞时会出现奇点，并乘上一组无自由矢量指标的超场。结合该信息、标量 (64) 与矢量构造块 (69) 的 BRST 变换性质，以及函数 $g^{(n)}(z,\tau)$ 和 ℓ^m 的单值性质，我们得到五点相关子的如下形式:

$$\mathcal{K}_5(\ell, z_i) = V_1 T^m_{2,3,4,5} \ell^m \tag{74}$$

$$+ V_{12} T_{3,4,5} g^{(1)}_{12} + (2 \leftrightarrow 3,4,5)$$

$$+ V_1 T_{23,4,5} g^{(1)}_{23} + (2,3 \mid 2,3,4,5) \ .$$

This is BRST invariant up to total worldsheet derivatives since

该相关子在整体世界面导数范围内满足 BRST 不变性，因为

$$Q\mathcal{K}_5(\ell, z_i)\mathcal{I}_5(\ell) = -V_1 V_2 T_{3,4,5}\left((\ell \cdot k_2) + s_{21}g_{21}^{(1)} + s_{23}g_{23}^{(1)}\right.$$
$$\left.+s_{24}g_{24}^{(1)} + s_{25}g_{25}^{(1)}\right)\mathcal{I}_5(\ell)$$
$$= -V_1 V_2 T_{3,4,5}\frac{\partial}{\partial z_2}\mathcal{I}_5(\ell), \tag{75}$$

where $\mathcal{I}_5(\ell)$ is the Koba-Nielsen factor (57).

其中 $\mathcal{I}_5(\ell)$ 是小林-尼尔森因子 (57)。

The correlator (74) is also homology invariant up to BRST-exact terms, around both A and B cycles as a function of ℓ^m and z_i . In other words, $\mathcal{K}_5(\ell, z_i)$ is an example of a generalized elliptic integrand [63]. To see this note that ℓ^m and $g_{ij}^{(n)}$ are single-valued around A cycles while $\ell^m \to \ell^m - 2\pi i k_j^m$ and $g_{ij}^{(1)} \to -2\pi i$ as z_j is moved around the B cycle (with $\tau \to \tau + 1$). That is, under the action of the monodromy operator (62), we get:

相关子 (74) 作为 ℓ^m 和 z_i 的函数，在 A 和 B 闭链周围也是同调不变的，至多相差 BRST 恰当项。换言之，$\mathcal{K}_5(\ell, z_i)$ 是广义椭圆被积函数的一个例子 [63]。不难发现，ℓ^m 和 $g_{ij}^{(n)}$ 在 A 闭链周围是单值的，而当 z_j 绕 B 闭链运动 (固定 $\tau \to \tau + 1$) 时，$\ell^m \to \ell^m - 2\pi i k_j^m$ 和 $g_{ij}^{(1)} \to -2\pi i$ 会发生变换。也就是说，在单值变换算子 (62) 的作用下，我们得到:

$$D\mathcal{K}_5(\ell) = \Omega_1\left(k_1^m V_1 T_{2,3,4,5}^m + \left[V_{12}T_{3,4,5} + 2 \leftrightarrow 3,4,5\right]\right) \tag{76}$$
$$+\Omega_2\left(k_2^m V_1 T_{2,3,4,5}^m + V_{21}T_{3,4,5} + \left[V_1 T_{23,4,5} + 3 \leftrightarrow 4,5\right]\right)$$
$$+\left(2 \leftrightarrow 3,4,5\right),$$

which can be shown to be BRST-exact [62] as it is BRST closed and a local five-point expression.

由于它是 BRST 闭的且是定域五点表达式，可以证明它是 BRST 恰当的 [62]。

Six points Similar considerations of the zero-mode structure from the pure spinor prescription together with BRST and homology invariance were used to determine the six-point correlator at genus one in [64]. The result can be written as (As discussed in [64], there is a beautiful Lie-polynomial compact representation of higher-point genus-one correlators which reveals a common structure with genus-zero correlators and elucidates the combinatorics of (77). However, as the notation requires concepts such as the decreasing Lyndon factorization of words and Lie polynomials [73], we chose to omit it here for brevity.)

六点基于纯自旋 or 处方的零模结构分析，结合 BRST 不变性与同调不变性，文献 [64] 确定了亏格一的六点关联函数。结果可写为:(正如文献 [64] 所述，高亏格一关联函数存在优美的李多项式紧凑表示，该表示揭示了它与亏格零关联函数的共同结构，阐明了 (77) 的组合性质。但由于该表示需要用到字的递减林登分解、李多项式等概念 [73]，为简洁起见，我们在此略去相关内容。)

$$\mathcal{K}_6\left(\ell, z_i\right)=\frac{1}{2} V_1 T_{2,3,4,5,6}^{mn} \mathcal{I}_{1,2,3,4,5,6}^{mn} \tag{77}$$

$$+V_{12} T_{3,4,5,6}^{m} \mathcal{Z}_{12,3,4,5,6}^{pm}+(2 \leftrightarrow 3,4,5,6)$$

$$+V_1 T_{23,4,5,6}^{m} \mathcal{L}_{1,23,4,5,6}^{m}+(2,3 \mid 2,3,4,5,6)$$

$$+V_{123} T_{4,5,6} \mathcal{L}_{123,4,5,6}+V_{132} T_{4,5,6} \mathcal{L}_{132,4,5,6}+(2,3 \mid 2,3,4,5,6)$$

$$+V_1 T_{234,5,6} \mathcal{L}_{1,234,5,6}+V_1 T_{243,5,6} \mathcal{L}_{1,243,5,6}+(2,3,4 \mid 2,3,4,5,6)$$

$$+\left[\left(V_{12} T_{34,5,6} \mathcal{L}_{12,34,5,6}+\operatorname{cyc}(2,3,4)\right)+(2,3,4 \mid 2,3,4,5,6)\right]$$

$$+\left[\left(V_1 T_{2,34,56} \mathcal{L}_{1,2,34,56}+\operatorname{cyc}(3,4,5)\right)+(2 \leftrightarrow 3,4,5,6)\right],$$

where the shorthand for the worldsheet functions are

其中世界面函数的简写为

$$\mathcal{Z}_{123,4,5,6}=g_{12}^{(1)} g_{23}^{(1)}+g_{12}^{(2)}+g_{23}^{(2)}-g_{13}^{(2)}, \tag{78}$$

$$\mathcal{Z}_{12,34,5,6}=g_{12}^{(1)} g_{34}^{(1)}+g_{13}^{(2)}+g_{24}^{(2)}-g_{14}^{(2)}-g_{23}^{(2)},$$

$$\mathcal{L}_{12,3,4,5,6}^{m}=\ell^m g_{12}^{(1)}+\left(k_2^m-k_1^m\right) g_{12}^{(2)}+\left[k_3^m\left(g_{13}^{(2)}-g_{23}^{(2)}\right)+(3 \leftrightarrow 4,5,6)\right],$$

$$\mathcal{Z}_{1,2,3,4,5,6}^{mn}=\ell^m \ell^n+\left[\left(k_1^m k_2^n+k_1^n k_2^m\right) g_{12}^{(2)}+(1,2 \mid 1,2,3,4,5,6)\right].$$

After a lenghty calculation, the correlator (77) was shown to be homology invariant up to vanishing BRST-exact terms, therefore constituting a six-point example of a generalized elliptic integrand. The analysis of BRST invariance is more subtle as the six-point open-string correlator at genus one is anomalous before summing over the different worldsheet topologies including the Möbius strip [49,50]. Since gauge invariance is reflected on BRST invariance, to study anomalous correlators, the concept of BRST pseudo-invariance was introduced in [70]. The idea is that the BRST variation of pseudo-invariant superfields generate anomalous superfields:

经过冗长计算，关联函数 (77) 被证明在相差一个消失的 BRST 恰当项的意义下满足同调不变性，因此它是广义椭圆被积函数的六点实例。BRST 不变性的分析更为微妙: 在对包括默比乌斯带在内的不同世界面拓扑求和之前，亏格一的六点开弦关联函数存在反常 [49,50]。由于规范不变性体现为 BRST 不变性，为研究反常关联函数，文献 [70] 引入了 BRST 伪不变性的概念。其核心思想是，伪不变超场的 BRST 变分生成反常超场:

$$Y_{A,B,C,D,E} = \frac{1}{2}(\lambda\gamma^m W_A)(\lambda\gamma^n W_B)(\lambda\gamma^p W_C)\big(W_D\gamma_{mnp}W_E\big) \tag{79}$$

generalizing the pure spinor superspace expression found in the six-point anomaly analysis of [28], $(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W$
, with parity-odd component expansion:

推广文献 [28] 六点反常分析中得到的纯自旋 or 超空间表达式 $(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)\big(W_5\gamma_{mnp}W_6\big)$，给出宇称-奇分量展开:

$$\big\langle(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)\big(W_5\gamma_{mnp}W_6\big)\big\rangle = -\frac{1}{16}\varepsilon_{10}^{m_2n_2\ldots m_6n_6}F^2_{m_2n_2}\ldots F^6_{m_6n_6}.$$

(80)

This is captured by the correlator (77) as its BRST variation, after discarding total worldsheet derivatives, is given by

这一点可由关联函数 (77) 体现: 丢弃世界面全导数后，其 BRST 变分可写为

$$Q\mathcal{K}_6(\ell,z_i)\,\mathcal{I}_6(\ell) = -2\pi i V_1 Y_{2,3,4,5,6}\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell)\,. \tag{81}$$

Thus, the BRST variation is a boundary term in moduli space [37] and vanishes due to the anomaly cancellation effect of summing over the different worldsheet topologies when the gauge group is $SO(32)$ [49, 50]
.

因此，当规范群为 $SO(32)$ [49, 50] 时，BRST 变分是模空间中的边界项，且由于对不同世界面拓扑求和的反常抵消效应而变为零。

Seven points A seven-point open-string correlator at genus one was also obtained in [64] and can be written using various kinds of generalized elliptic integrands $E_{\cdots|\cdots}$ discussed at length in [63]:

七点文献 [64] 也得到了亏格一的七点开弦关联函数，它可以用 [63] 中详细讨论的各类广义椭圆被积函数 $E_{\cdots|\cdots}$ 表示:

$$\mathcal{K}_7(\ell,z_i) = \frac{1}{6}V_1 T^{mnp}_{2,3,\ldots,7}E^{mnp}_{1|2,3,\ldots,7} \tag{82}$$

$$+\frac{1}{2}V_1 T^{mn}_{23,4,5,6,7}E^{mn}_{1|23,4,5,6,7} + (2,3\,|\,2,3,4,5,6,7)$$

$$+\big[V_1 T^{m}_{234,5,6,7}E^{m}_{1|234,5,6,7} + V_1 T^{m}_{243,5,6,7}E^{m}_{1|243,5,6,7}\big]$$

$$+ (2,3,4 \mid 2,3,4,5,6,7)$$

$$+ \left[V_1 T^m_{23,45,6,7} E^m_{1|23,45,6,7} + \mathrm{cyc}\,(2,3,4)\right] + (6,7 \mid 2,3,4,5,6,7)$$

$$+ \left[V_1 T_{2345,6,7} E_{1|2345,6,7} + \mathrm{perm}\,(3,4,5)\right] + (2,3,4,5 \mid 2,3,4,5,6,7)$$

$$+ \left[V_1 T_{234,56,7} E_{1|234,56,7} + V_1 T_{243,56,7} E_{1|243,56,7} + \mathrm{cyc}\,(5,6,7)\right]$$

$$+ (2,3,4 \mid 2,3,4,5,6,7)$$

$$+ \left[V_1 T_{23,45,67} E_{1|23,45,67} + \mathrm{cyc}\,(4,5,6)\right] + (3 \leftrightarrow 4,5,6,7)$$

$$- V_1 J^m_{2|3,4,5,6,7} E^m_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)$$

$$- V_1 J_{23|4,5,6,7} E_{1|23|4,5,6,7} + (2,3 \mid 2,3,4,5,6,7)$$

$$- \left[V_1 J_{2|34,5,6,7} E_{1|2|34,5,6,7} + \mathrm{cyc}\,(2,3,4)\right] + (2,3,4 \mid 2,3,4,5,6,7)\,.$$

This was shown to be BRST (pseudo) invariant and also homology invariant up to BRST-exact terms and total derivatives in the worldsheet and in moduli space.

该关联函数被证明满足 BRST(伪) 不变性，且在相差 BRST 恰当项、世界面和模空间全导数的意义下满足同调不变性。

One-Loop SYM Integrands from the Cohomology of Pure Spinor Superspace

来自纯自旋手超空间上同调的单圈 SYM 被积函数

Another application of the pure spinor formalism and related ideas resulted in expressions for the one-loop integrands of ten-dimensional SYM theory [65]. The idea is to use the zero-mode structure suggested by the pure spinor prescription, i.e., after removing nonzero modes via OPEs leading to multiparticle superfields, to directly propose SYM one-loop integrands $A(1,2,\ldots,n \mid \ell)$ governing the integrated single-trace amplitude via

纯自旋手形式论及相关思想的另一项应用给出了十维 SYM 理论的单圈被积函数表达式 [65]。其核心思路是利用纯自旋手方案给出的零模结构: 即通过算符乘积展开 OPE 消去非零模得到多粒子超场后，直接给出描述积分单迹振幅的 SYM 单圈被积函数 $A(1,2,\ldots,n \mid \ell)$，其形式为

$$A(1,2,3,\ldots,n) = \int \frac{d^D \ell}{(2\pi)^D} \langle A(1,2,3,\ldots,n \mid \ell)\rangle. \tag{83}$$

More precisely, the one-loop integrands are expanded in terms of cubic graphs Γ_i :

更准确地说，单圈被积函数可按三次图 Γ_i 展开:

$$A(1,2,3,\ldots,n \mid \ell) = \sum_{\Gamma_i} \frac{N_i(\ell)}{\prod_k P_{k,i}(\ell)}, \tag{84}$$

where the sum is over all one-loop cubic graphs from boxes to n -gons, excluding triangles, bubbles, and tadpoles [30]. Note that the superspace numerators $N_i(\ell)$ and the propagators $P_{k,i}(\ell)$ depend not only on the external kinematics but also on the loop momentum ℓ . In proposing the integrand (84), one respects the supersymmetry constraint that the numerators of a p -gon diagram contain at most $p-4$ powers of ℓ. Furthermore, it is not difficult to be convinced that overall BRST invariance of the integrand can be achieved only if each term of $QN_i(\ell)$ has a factor of $P_{k,i}(\ell)$ with $k=1,2,\ldots,n$. Schematically,

求和遍历从箱图到 n 边形的所有单圈三次图，排除三角形、泡泡图和蝌蚪图 [30]。注意超空间分子 $N_i(\ell)$ 和传播子 $P_{k,i}(\ell)$ 不仅依赖外部运动学，还依赖圈动量 ℓ 。在构造被积函数 (84) 时，需要满足超对称约束: p 边形图的分子最多包含 $p-4$ 次 ℓ 幂。此外不难发现，只有当 $QN_i(\ell)$ 的每一项都带一个满足 $k=1,2,\ldots,n$ 的因子 $P_{k,i}(\ell)$ 时，才能得到整体 BRST 不变的被积函数。概略地写为,

$$QN_i(\ell) = \sum P_{k,i}(\ldots), \tag{85}$$

for some subset of k with the ellipsis representing combinations of (multiparticle) superfields. Integrands up to six points were found in [65] following these lines.

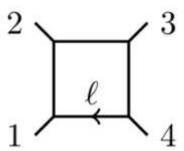
其中针对 k 的某个子集，省略号代表 (多粒子) 超场的组合。按照这一思路，文献 [65] 已经得到了六点及以下的被积函数。

Four-point integrand The integrand of the color-ordered amplitude is expressed in terms of a single box with a BRST-invariant numerator:

四点被积函数: 色序振幅的被积函数可以表示为一个带 BRST 不变分子的单箱图:

(86)

$$A(1,2,3,4|\ell) =$$



$$= \frac{V_1 T_{2,3,4}}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2}.$$

This integrand is manifestly BRST invariant using (66) and agrees with the result obtained by Green, Brink, and Schwarz [57] from the field-theory limit of string theory.

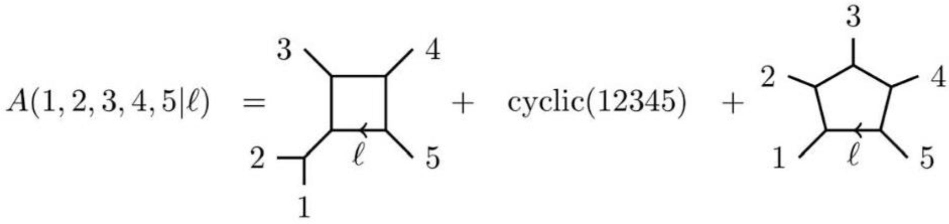
利用 (66) 可证该被积函数显然满足 BRST 不变性，且与 Green、Brink 和 Schwarz 从弦论场论极限得到的结果一致 [57]。

Five-point integrand The integrand of the SYM five-point one-loop amplitude is expanded in terms of five boxes and one pentagon:

五点被积函数:SYM 五点单圈振幅的被积函数可展开为五个箱图加一个五边形图:

(87)

$$
= A_{\text{box}}(1,2,3,4,5) + A_{\text{pent}}(1,2,3,4,5 \mid \ell)
$$



with the corresponding pure spinor superspace expressions given by

对应的纯自旋手超空间表达式由下式给出

$$
\begin{aligned}
A_{\text{box}}(1,2,3,4,5) = & \frac{V_{12} T_{3,4,5}}{(k_1+k_2)^2 \ell^2 (\ell-k_{12})^2 (\ell-k_{123})^2 (\ell-k_{1234})^2} \\
& + \frac{V_1 T_{23,4,5}}{(k_2+k_3)^2 \ell^2 (\ell-k_1)^2 (\ell-k_{123})^2 (\ell-k_{1234})^2} \\
& + \frac{V_1 T_{2,34,5}}{(k_3+k_4)^2 \ell^2 (\ell-k_1)^2 (\ell-k_{12})^2 (\ell-k_{123})^2} \\
& + \frac{V_1 T_{2,3,45}}{(k_4+k_5)^2 \ell^2 (\ell-k_1)^2 (\ell-k_{12})^2 (\ell-k_{123})^2} \\
& + \frac{V_{51} T_{2,3,4}}{(k_1+k_5)^2 (\ell-k_1)^2 (\ell-k_{12})^2 (\ell-k_{123})^2 (\ell-k_{1234})^2}
\end{aligned} \tag{88}
$$

$$
A_{\text{pent}}(1,2,3,4,5 \mid \ell) = \frac{N^{(5)}_{1|2,3,4,5}(\ell)}{\ell^2 (\ell-k_1)^2 (\ell-k_{12})^2 (\ell-k_{123})^2 (\ell-k_{1234})^2} \tag{89}
$$

and pentagon numerator

以及五边形分子

$$
\begin{aligned}
N^{(5)}_{1|2,3,4,5}(\ell) = & \ell_m V_1 T^m_{2,3,4,5} + \frac{1}{2}\left[V_{12} T_{3,4,5} + (2 \leftrightarrow 3,4,5)\right] \\
& + \frac{1}{2}\left[V_1 T_{23,4,5} + (2,3 \mid 2,3,4,5)\right].
\end{aligned} \tag{90}
$$

To see that this integrand is BRST invariant, note that the BRST variation of the local pentagon numerator satisfies the criterion of canceling pentagon propagators as

要证明该被积函数是 BRST 不变的，注意局域五边形分子的 BRST 变分满足抵消五边形传播子的条件，即

$$QN^{(5)}_{1|2,3,4,5}(\ell) = \frac{1}{2}V_1V_2T_{3,4,5}\left[(\ell-k_{12})^2-(\ell-k_1)^2\right] \tag{91}$$

$$+\frac{1}{2}V_1V_3T_{2,4,5}\left[(\ell-k_{123})^2-(\ell-k_{12})^2\right]$$

$$+\frac{1}{2}V_1V_4T_{2,3,5}\left[(\ell-k_{1234})^2-(\ell-k_{123})^2\right]$$

$$+\frac{1}{2}V_1V_5T_{2,3,4}\left[\ell^2-(\ell-k_{1234})^2\right]$$

implies that $QA_{\text{pent}}(1,2,3,4,5\mid\ell)$ becomes a sum of boxes of the same type as contained in $A_{\text{box}}(1,2,3,4,5)$. In turn, the BRST variation of the boxes cancels the external propagators with external momenta k_i in (88) rather than the internal propagators with loop momentum. Therefore, the BRST variation of $QA_{\text{box}}(1,2,3,4,5)$ is still a sum of boxes:

这意味着 $QA_{\text{pent}}(1,2,3,4,5\mid\ell)$ 可化为与 $A_{\text{box}}(1,2,3,4,5)$ 中同类型箱图的和。反过来，箱图的 BRST 变分会抵消 (88) 中带外部动量 k_i 的外部传播子，而非带圈动量的内部传播子。因此，$QA_{\text{box}}(1,2,3,4,5)$ 的 BRST 变分仍然是箱图的和:

$$QA_{\text{box}}(1,2,3,4,5) = \frac{V_1V_2T_{3,4,5}}{2\ell^2(\ell-k_{123})^2(\ell-k_{1234})^2}\left(\frac{1}{(\ell-k_{12})^2}-\frac{1}{(\ell-k_1)^2}\right)$$

(92)

$$+\frac{V_1V_3T_{2,4,5}}{2\ell^2(\ell-k_1)^2(\ell-k_{1234})^2}\left(\frac{1}{(\ell-k_{123})^2}-\frac{1}{(\ell-k_{12})^2}\right)$$
$$+\frac{V_1V_4T_{2,3,5}}{2\ell^2(\ell-k_1)^2(\ell-k_{12})^2}\left(\frac{1}{(\ell-k_{1234})^2}-\frac{1}{(\ell-k_{123})^2}\right)$$
$$+\frac{V_1V_5T_{2,3,4}}{2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2}\left(\frac{1}{\ell^2}-\frac{1}{(\ell-k_{1234})^2}\right).$$

which ultimately cancels the variation of the pentagon (89), leading to an overall BRST-invariant five-point one-loop integrand. This example illustrates the mechanism that the BRST variation of a numerator must be engineered to cancel either internal or external propagators in order to achieve overall BRST invariance.

它最终会抵消五边形 (89) 的变分，从而得到整体 BRST 不变的五点单圈被积函数。这个例子阐明了实现整体 BRST 不变性的机制: 必须将分子的 BRST 变分构造为抵消内部或外部传播子的形式。

Six-point integrand The 6-point integrand is composed of 21 boxes, 6 pentagons, and 1 hexagon:

六点被积函数: 六点被积函数由 21 个箱图、6 个五边形图和 1 个六边形图构成:

$$A\left(1,2,\ldots,6 \mid \ell\right)=A_{\text{box}}\left(1,2,\ldots,6\right)+A_{\text{pent}}\left(1,2,\ldots,6 \mid \ell\right)+A_{\text{hex}}\left(1,2,\ldots,6 \mid \ell\right),$$

(93)

whose superspace expressions can be found in [65]. A noteworthy feature of the pure spinor superspace proposal for (93) is that it leads to an anomalous integrated BRST variation:

其超空间表达式可在文献 [65] 中找到。(93) 的纯自旋手超空间构造有一个值得注意的特点: 它会给出反常的积分 BRST 变分:

$$\int d^D\ell QA\left(1,2,3,4,5,6 \mid \ell\right)=-\frac{\pi^5}{240}V_1Y_{2,3,4,5,6}, \tag{94}$$

signaling the well-known fact that the ten-dimensional SYM theory is anomalous at one loop; see [65] for more details.

这也印证了十维 SYM 理论在单圈存在反常这一共识，更多细节参见 [65]。

Note that the six-point one-loop integrand was recently derived in [33] in a parameterization satisfying the one-loop color-kinematics duality.

请注意，六点单圈被积函数近期已在文献 [33] 中，以满足单圈色-运动学对偶性的参数化形式推导得出。

Genus Two

亏格二

After the pioneering genus-two calculation with four massless NS states with the RNS formalism in [40], the pure spinor formalism was used in [22, 23] to extend the computation to the supersymmetric graviton multiplet (see also [24,45,68] for explicit component expansions and the overall normalization factor). For five massless closed-string states, the supersymmetric amplitudes were computed in the low-energy approximation including their overall normalization in [48] and later extended to all orders in α' in [41, 42] .

继文献 [40] 中利用 RNS 形式完成开创性的四个无质量 NS 态亏格二计算后，文献 [22, 23] 利用纯自旋子形式将计算推广到超对称引力子多重态 (显分量展开与整体归一化因子另见文献 [24,45,68])。对于五个无质量闭弦态，文献 [48] 在低能近似下计算了超对称振幅 (含整体归一化)，之后在 [41, 42] 中将其推广到 α' 的所有阶。

The n -point amplitude prescription (27) gives rise to a chiral amplitude $\mathcal{F}_n$ which factorizes into a Koba-Nielsen factor (in conventions where $s_{ij}=k_i\cdot k_j$):

n 点振幅规则 (27) 给出了手征振幅 $\mathcal{F}_n$ ，该振幅可因子化为 Koba-Nielsen 因子 (在 $s_{ij}=k_i\cdot k_j$ 的规范下):

$$\mathcal{J}_n = \exp\left(\frac{1}{4\pi i}\Omega_{IJ}\ell^I\cdot\ell^J - \sum_{i=1}^{n}\left(\ell^I\cdot k_i\right)\int_{z_0}^{z_i}\omega_I + \sum_{i<j}^{n} s_{ij}\ln E\left(z_i,z_j\right)\right) \tag{95}$$

and a chiral correlator $\mathcal{K}_n\left(\ell^I, z_i\right)$ carrying the dependence on loop momenta, vertex operator positions, and the polarizations and external momenta of the string states. Since the vertex positions will be integrated over the Riemann surface, one is free to use chiral correlators which differ by total derivatives as representing the same amplitude under integration by parts (IBP). For instance, the logarithmic derivative of the Koba-Nielsen factor is a primary example of an IBP generator:

以及一个携带圈动量、顶点算子位置、弦态极化和外动量依赖关系的手征关联函数 $\mathcal{K}_n\left(\ell^I, z_i\right)$。由于顶点位置需要在黎曼曲面上积分，分部积分 (IBP) 下，相差全导数的手征关联函数可以自由地用来表示同一个振幅。例如，Koba-Nielsen 因子的对数导数就是 IBP 生成元的典型例子:

$$\partial_{z_1}\ln\mathcal{J}_5 = -\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right) + s_{12}\eta_{12} + s_{13}\eta_{13} + s_{14}\eta_{14} + s_{15}\eta_{15}. \tag{96}$$

BRST invariance The integration of the zero modes of pure spinor fields together with considerations from group theory to piece together Lorentz-invariant combinations of superfields initially led to the introduction of pure spinor superfield building blocks with four external states in [24]:

BRST 不变性最初，为了对纯自旋子场零模积分，并结合群论考虑拼出超场的洛伦兹不变组合，文献 [24] 引入了四个外态的纯自旋子超场构造块:

$$T_{1,2|3,4} = \frac{1}{64}\left(\lambda\gamma_{\text{mnpqr}}\lambda\right)F_1^{mn}F_2^{pq}\left[F_3^{rs}\left(\lambda\gamma_s W_4\right) + F_4^{rs}\left(\lambda\gamma_s W_3\right)\right] + (1,2\leftrightarrow 3,4) \tag{97}$$

Considerations involving the BRST variations of suitable multiparticle superfields allowed an all-multiplicity generalization of (97) to be found in [69]. Using the language of (minimal) pure spinor superspace, these generalizations have the form [69]:

通过对合适的多粒子超场 BRST 变分的研究，文献 [69] 找到了 (97) 对任意多重度的推广。利用 (最小) 纯自旋子超空间语言，这些推广的形式为 [69]:

$$T_{A,B|C,D} = \frac{1}{64}\left(\lambda\gamma_{\text{mnpqr}}\lambda\right)F_A^{mn}F_B^{pq}\left[F_C^{rs}\left(\lambda\gamma_s W_D\right) + F_D^{rs}\left(\lambda\gamma_s W_C\right)\right] + (A,B\leftrightarrow C,D) \tag{C,D}$$

(98)

In addition, starting from the five-point correlator, there are additional Lorentz scalar and tensorial building blocks:

此外，从五点关联函数开始，还存在额外的洛伦兹标量与张量构造块:

$$T^m_{1,2,3|4,5} = A_1^m T_{2,3|4,5} + A_2^m T_{1,3|4,5} + A_3^m T_{1,2|4,5} + W^m_{1,2,3|4,5}, \tag{99}$$

$$T_{1;2|3|4,5} = \frac{1}{2}\left(\left(k_1^m + k_2^m - k_3^m\right)T^m_{1,2,3|4,5} + T_{12,3|4,5} + T_{13,2|4,5} + T_{23,1|4,5}\right),$$

where $W^m_{1,2,3|4,5}$ is designed in a way as to give the desired BRST variation and symmetry properties to $T^m_{1,2,3|4,5}$; see below. Its explicit form can be found in [69]. The BRST variation of the four-point building block is given by

其中 $W^m_{1,2,3|4,5}$ 的构造满足使 $T^m_{1,2,3|4,5}$ 获得预期 BRST 变分与对称性性质的要求，见下文。其显式形式可在文献 [69] 中找到。四点构造块的 BRST 变分由下式给出

$$QT_{1,2|3,4} = 0 \tag{100}$$

while the BRST variation of the five-point building blocks is given by

而五点构造块的 BRST 变分由下式给出

$$QT_{12,3|4,5} = s_{12}\left(V_1 T_{2,3|4,5} - V_2 T_{1,3|4,5}\right), \tag{101}$$

$$QT_{1;2|3|4,5} = s_{12} V_1 T_{2,3|4,5},$$

$$QT^m_{1,2,3|4,5} = k^m_1 V_1 T_{2,3|4,5} + k^m_2 V_2 T_{1,3|4,5} + k^m_3 V_3 T_{1,2|4,5}.$$

Furthermore, these building blocks satisfy various crucial identities to capture the correct features of the integrand:

此外，这些构造块满足多个关键恒等式，以刻画被积函数的正确性质:

$$T_{A,B|C,D} = T_{B,A|C,D} = T_{C,D|A,B},\ T_{A,B|C,D} + T_{B,C|A,D} + T_{C,A|B,D} = 0$$

(102)

$$T^m_{1,2,3|4,5} = T^m_{(1,2,3)|(4,5)},\ \langle T^m_{1,2,3|4,5}\rangle = \langle T^m_{3,4,5|1,2} + T^m_{2,4,5|1,3} + T^m_{1,4,5|2,3}\rangle$$

$$T_{1;2|3|4,5} = T_{1;2|3|5,4},\ \langle T_{1;2|3|4,5} + T_{1;2|4|5,3} + T_{1;2|5|3,4}\rangle = 0$$

$$\langle k^m_3\left(T^m_{1,2,3|4,5} + T^m_{3,4,5|1,2}\right) - T_{13,2|4,5} - T_{23,1|4,5} + T_{34,5|1,2} + T_{35,4|1,2}\rangle = 0$$

$$k^m_1 T^m_{1,2,3|4,5} = T_{2;1|3|4,5} + T_{3;1|2|4,5}$$

$$\langle k^m_5 T^m_{1,2,3|4,5}\rangle = \langle T_{1;5|4|2,3} + T_{2;5|4|1,3} + T_{3;5|4|1,2}\rangle$$

$$T_{1;2|3|4,5} - T_{2;1|3|4,5} = T_{12,3|4,5}$$

$$\langle T_{5;1|2|3,4} + T_{5;2|1|3,4} + T_{5;3|4|1,2} + T_{5;4|3|1,2}\rangle = 0$$

where the identities that hold only in the cohomology have been indicated by the pure spinor bracket.

其中仅上同调中成立的恒等式已用纯自旋子括号标出。

Homology invariance The genus-two integrands up to five points can be written in terms of the holomorphic differentials ω_I and loop momenta ℓ_I^m , the prime form $E\left(z_i, z_j\right)$, and single derivatives of its logarithm $\partial_i \ln E\left(z_i, z_j\right)$. Note that the prime form $E(z, w)$ is holomorphic in z and w , odd under $z \leftrightarrow w$, and has a unique simple zero at $z = w$. Both the loop momentum and the prime form are single-valued when z is moved around A_I cycles, but they have nontrivial monodromy around a B_I cycle [39]:

同调不变性直到五点的亏格二被积函数都可以用全纯微分 ω_I 、圈动量 ℓ_I^m 、素形式 $E\left(z_i, z_j\right)$ 及其对数的单导数 $\partial_i \ln E\left(z_i, z_j\right)$ 表示。注意素形式 $E(z, w)$ 在 z 和 w 中全纯，在 $z \leftrightarrow w$ 变换下为奇，且在 $z = w$ 处有唯一单零点。当 z 绕 A_I 圈运动时，圈动量和素形式都是单值的，但它们绕 B_I 圈运动时具有非平凡单值性 [39]:

$$E\left(z + B_I, w\right) = -\exp\left(-i\pi\Omega_{II} - 2\pi i\int_w^z \omega_I\right) E(z, w) \tag{103}$$

$$\partial_z \ln E\left(z + B_I, w\right) = \partial_z \ln E(z, w) - 2\pi i \omega_I(z)$$

$$\partial_z \ln E\left(z, w + B_I\right) = \partial_z \ln E(z, w) + 2\pi i \omega_I(z)$$

$$\ell_I^m = \ell_I^m - 2\pi i k_i^m$$

In order to avoid clutter in the formulas below, it is convenient to define the genus-two propagator as

为了避免后续公式过于繁杂，conveniently 将亏格二传播子定义为

$$\eta_{ij} = \frac{\partial}{\partial z_i} \ln E\left(z_i, z_j\right). \tag{104}$$

Basis of holomorphic one-forms At genus two the holomorphic one-forms $\omega_I\left(z_i\right)$ are labeled by $I = 1, 2$ and modular invariance of the string amplitude suggests that they form $SL(2)$ - invariant singlets (Discussions with Oliver Schlotterer are warmly acknowledged at this point.), where $\omega_I(z)$ is the (1) of $SL(2)$. At four points, the tensor decomposition $(1) \otimes (1) \otimes (1) \otimes (1) = 2(0) \oplus 3(2) \oplus (4)$ [77] shows that there are two scalars in the decomposition of a fourfold product of $\omega_I\left(z_i\right)$. Using the definition

全纯一维形式基在亏格二的情况下，全纯一维形式 $\omega_I\left(z_i\right)$ 由 $I = 1, 2$ 标记，弦振幅的模不变性表明它们构成 $SL(2)$ 不变单态 (在此特别感谢与 Oliver Schlotterer 的讨论)，其中 $\omega_I(z)$ 是 $SL(2)$ 的 (1)。在四点情况下，张量分解 $(1) \otimes (1) \otimes (1) \otimes (1) = 2(0) \oplus 3(2) \oplus (4)$ [77] 表明，$\omega_I\left(z_i\right)$ 四重积的分解中存在两个标量。利用定义

$$\Delta_{ij} = \varepsilon^{IJ} \omega_I\left(z_i\right) \omega_J\left(z_j\right) \tag{105}$$

the two-dimensional basis of scalars composed of four holomorphic one-forms can be taken in a cyclic arrangement of labels:

由四个全纯一维形式构成的二维标量基可通过标签的循环排列得到:

$$\Delta_{12}\Delta_{34},\ \Delta_{23}\Delta_{41}. \tag{106}$$

A third scalar $\Delta_{13}\Delta_{24}$ is not independent as the antisymmetrization over three indices vanishes:

第三个标量 $\Delta_{13}\Delta_{24}$ 不独立，因为对三个指标的反对称化等于零:

$$0 = \varepsilon^{I\left[J\varepsilon^{KL}\right]}\omega_I(z_1)\,\omega_J(z_2)\,\omega_K(z_3)\,\omega_L(z_4) \to \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} - \Delta_{41}\Delta_{23}.$$

(107)

At five points, the decomposition $(1)\otimes(1)\otimes(1)\otimes(1)\otimes(1) = 5(1)\oplus 4(3)\oplus(5)$ shows that there exists a five-dimensional basis of a fivefold product of one-forms. These can be taken in a cyclic basis [41] (An algorithm to arrive at this basis uses the two identities $\Delta_{ij}\Delta_{kl} = -\Delta_{il}\Delta_{kj} - \Delta_{ij}\Delta_{lk}$ and $\omega_I(z_i)\,\Delta_{jk} = -\omega_I(z_k)\,\Delta_{ij} - \omega_I(z_j)\,\Delta_{ki}$ repeatedly until all factors are in cyclic order. It can be shown that the two identities $\omega_I(z_1)\,\Delta_{24}\Delta_{35} = -\omega_I(z_5)\,\Delta_{12}\Delta_{34} - \omega_I(z_2)\,\Delta_{51}\Delta_{34} + \omega_I(z_1)\,\Delta_{23}\Delta_{45}$ and $\omega_I(z_1)\,\Delta_{25}\Delta_{34} = -\omega_I(z_5)\,\Delta_{12}\Delta_{34} - \omega_I(z_2)\,\Delta_{34}\Delta_{51}$ as well as their permutations are enough to rewrite all products of five one-forms in the basis (108).)

五点情况下，分解 $(1)\otimes(1)\otimes(1)\otimes(1)\otimes(1) = 5(1)\oplus 4(3)\oplus(5)$ 表明，一维形式五重积存在一个五维基。这些基可选取为循环基 [41](得到该基的算法反复利用两个恒等式 $\Delta_{ij}\Delta_{kl} = -\Delta_{il}\Delta_{kj} - \Delta_{ij}\Delta_{lk}$ 和 $\omega_I(z_i)\,\Delta_{jk} = -\omega_I(z_k)\,\Delta_{ij} - \omega_I(z_j)\,\Delta_{ki}$，直到所有因子都处于循环顺序。可以证明，两个恒等式 $\omega_I(z_1)\,\Delta_{24}\Delta_{35} = -\omega_I(z_5)\,\Delta_{12}\Delta_{34} - \omega_I(z_2)\,\Delta_{51}\Delta_{34} + \omega_I(z_1)\,\Delta_{23}\Delta_{45}$ 和 $\omega_I(z_1)\,\Delta_{25}\Delta_{34} = -\omega_I(z_5)\,\Delta_{12}\Delta_{34} - \omega_I(z_2)\,\Delta_{34}\Delta_{51}$ 以及它们的排列就足以将五个一维形式的所有乘积改写为基 (108) 的形式。)

$$\omega_I(z_1)\,\Delta_{23}\Delta_{45},\ \omega_I(z_2)\,\Delta_{34}\Delta_{51},\ \omega_I(z_3)\,\Delta_{45}\Delta_{12}, \tag{108}$$

$$\omega_I(z_4)\,\Delta_{51}\Delta_{23},\ \omega_I(z_5)\,\Delta_{12}\Delta_{34}.$$

Four points The massless four-point chiral integrand was obtained using the minimal pure spinor formalism in [23] and using the non-minimal formalism in [22]. Luckily, both versions of the formalism imply that the chiral integrand is obtained purely from the zero modes of pure spinor variables. A short analysis of the zero-mode structure of the contributing SYM superfields together with a group theory analysis of $SO(10)$ scalars in pure spinor superspace using a $U(5)$ decomposition of pure spinors implies [23]

四点无质量四点手征被积函数分别在 [23] 中用极小纯自旋形式、[22] 中用非极小纯自旋形式得到。幸运的是，两种形式都表明手征被积函数完全由纯自旋变量的零模得到。对贡献 SYM 超场的零模结构的简要分析，结合对纯自旋超空间中 $SO(10)$ 标量利用纯自旋 $U(5)$ 分解的群论分析，得到结论 [23]

$$\mathcal{K}_4 = \left\langle T_{1,2|3,4}\right\rangle \Delta_{41}\Delta_{23} + \left\langle T_{4,1|2,3}\right\rangle \Delta_{12}\Delta_{34} \tag{109}$$

where $T_{i,j|k,l}$ is the kinematic factor (97) in the minimal pure spinor formalism and Δ_{ij} is defined in (105). It is easy to see that the chiral correlator (109) is BRST closed using (101). Moreover, it is manifestly single-valued as it only depends on the vertex positions via Δ_{ij} .

其中 $T_{i,j|k,l}$ 是极小纯自旋形式下的运动学因子 (97)，Δ_{ij} 由式 (105) 定义。利用式 (101) 可轻易看出手征关联函数 (109) 是 BRST 闭的。此外，它显然是单值的，因为它仅通过 Δ_{ij} 依赖于顶点位置。

It was shown in [68] via pure spinor BRST cohomology identities that the genus-two kinematic factor (97) is proportional to the four-point tree amplitude (50):

文献 [68] 通过纯自旋 BRST 上同调恒等式证明，亏格二运动学因子 (97) 正比于四点树图振幅 (50):

$$\langle T_{1,2|3,4}\rangle = s_{12}^2 s_{23} A\,(1,2,3,4)\,. \tag{110}$$

Five points Several equivalent expressions for the five-point chiral integrand, emphasizing different properties, were given in [41]. For instance,

五点文献 [41] 给出了五点手征被积函数的数个等价表达式，分别侧重不同性质，例如:

$$\mathcal{K}_5\left(\ell^I, z_i\right) = \left[\ell_m^I T^m_{1,2,3|4,5}\Delta_{51}\omega_I\left(z_2\right)\Delta_{34} + \text{cycl}\,(1,2,3,4,5)\right]$$

(111)

$$+\left[\eta_{12}\left(T_{1;2|3|4,5}\Delta_{24}\Delta_{35} + T_{1;2|4|3,5}\Delta_{23}\Delta_{45}\right) + (1,2\mid 1,2,3,4,5)\right]$$

$$+\left[\eta_{21}\left(T_{2;1|3|4,5}\Delta_{14}\Delta_{35} + T_{2;1|4|3,5}\Delta_{13}\Delta_{45}\right) + (1,2\mid 1,2,3,4,5)\right]$$

where the notation $+(i,j\mid 1,2,3,4,5)$ means a sum over all ordered choices of i and j from the set $\{1,2,3,4,5\}$ for a total of $\begin{pmatrix}5\\2\end{pmatrix}$ terms.

其中记号 $+(i,j\mid 1,2,3,4,5)$ 表示对从集合 $\{1,2,3,4,5\}$ 中有序选取 i 和 j 的所有情况求和，总共有 $\begin{pmatrix}5\\2\end{pmatrix}$ 项。

BRST invariance Using the BRST variation (101) of the building blocks, the BRST variation of the chiral correlator (111) can be written as

BRST 不变性利用构造块的 BRST 变分 (101)，手征关联函数 (111) 的 BRST 变分可写为

$$Q\mathcal{K}_5\left(\ell^I, z_i\right) = V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45}\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right)$$

(112)

$$+V_1 T_{2,3|4,5}\Delta_{12}\Delta_{34}\left(\ell^I\cdot k_1\right)\omega_I\left(z_5\right)$$

$$+V_1 T_{2,3|4,5}\Delta_{51}\Delta_{34}\left(\ell^I\cdot k_1\right)\omega_I\left(z_2\right)$$

$$+V_1\left(T_{2,3|4,5}\Delta_{24}\Delta_{35}+T_{2,4|3,5}\Delta_{23}\Delta_{45}\right)s_{12}\eta_{12}$$

$$+V_1\left(T_{3,2|4,5}\Delta_{34}\Delta_{25}+T_{3,4|2,5}\Delta_{32}\Delta_{45}\right)s_{13}\eta_{13}$$

$$+V_1\left(T_{4,2|3,5}\Delta_{43}\Delta_{25}+T_{4,3|2,5}\Delta_{42}\Delta_{35}\right)s_{14}\eta_{14}$$

$$+V_1\left(T_{5,2|3,4}\Delta_{53}\Delta_{24}+T_{5,3|2,4}\Delta_{52}\Delta_{34}\right)s_{15}\eta_{15}+\operatorname{cyc}(1,2,3,4,5)$$

To see that the terms proportional to V_1 are zero up to a total derivative with respect to z_1 , after replacing

为了说明正比于 V_1 的项在替换后，对 z_1 而言只差一个全导数而为零，

$$\Delta_{12}\Delta_{34}\left(\ell^I\cdot k_1\right)\omega_I\left(z_5\right)=-\Delta_{51}\Delta_{34}\left(\ell^I\cdot k_1\right)\omega_I\left(z_2\right)-\Delta_{25}\Delta_{34}\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right)$$

(113)

the terms containing the loop momenta simplify to

包含圈动量的项可化简为

$$\left(V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45}-V_1 T_{2,3|4,5}\Delta_{25}\Delta_{34}\right)\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right)\cong \tag{114}$$

$$\left(V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45}-V_1 T_{2,3|4,5}\Delta_{25}\Delta_{34}\right)\left(s_{12}\eta_{12}+s_{13}\eta_{13}+s_{14}\eta_{14}+s_{15}\eta_{15}\right)$$

where the IBP relation (96) $-\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right)+s_{12}\eta_{12}+s_{13}\eta_{13}+s_{14}\eta_{14}+s_{15}\eta_{15}\cong 0$ has been used. Plugging this into (112), the terms containing $s_{12}\eta_{12}$ become:

这里已经用到了 IBP 关系 (96) $-\left(\ell^I\cdot k_1\right)\omega_I\left(z_1\right)+s_{12}\eta_{12}+s_{13}\eta_{13}+s_{14}\eta_{14}+s_{15}\eta_{15}\cong 0$。将其代入式 (112)，包含 $s_{12}\eta_{12}$ 的项变为:

$$s_{12}\eta_{12}V_1\left(T_{2,3|4,5}\left(\Delta_{24}\Delta_{35}-\Delta_{25}\Delta_{34}\right)+\left(T_{2,4|3,5}+T_{2,5|3,4}\right)\Delta_{23}\Delta_{45}\right)=0 \tag{115}$$

where we used the kinematic Jacobi identity $T_{2,4|3,5}+T_{2,5|3,4}=-T_{2,3|4,5}$ as well as the worldsheet Jacobi identity $\Delta_{24}\Delta_{35}-\Delta_{25}\Delta_{34}-\Delta_{23}\Delta_{45}=0$. The analysis of the other terms $s_{1j}\eta_{1j}$ for $j=3,4,5$ is similar, and the vanishing of the full BRST variation (112) follows from the cyclic permutations.

其中我们用到了运动学雅可比恒等式 $T_{2,4|3,5}+T_{2,5|3,4}=-T_{2,3|4,5}$ 和世界面雅可比恒等式 $\Delta_{24}\Delta_{35}-\Delta_{25}\Delta_{34}-\Delta_{23}\Delta_{45}=0$。对 $j=3,4,5$ 其余项 $s_{1j}\eta_{1j}$ 的分析是类似的，整个 BRST 变分 (112) 为零可由循环排列得到。

Homology invariance Using the monodromies in (103), one can show that the chiral correlator (111) is single-valued as a function of ℓ_I and z_i . For instance, moving z_1 around the B_I cycle and writing the result in terms of the cyclic basis (108) implies that (111) is single-valued around z_1 provided

同调不变性利用 (103) 中的单值群，可以证明手征关联函数 (111) 作为 ℓ_I 和 z_i 的函数是单值的。例如，当 z_1 绕 B_I 圈移动后，将结果用循环基 (108) 表示，可得只要满足下式，(111) 在 z_1 附近就是单值的

$$\left\langle k_1^m T^m_{3,4,5|1,2} - T_{3;1|2|4,5} - T_{4;1|2|3,5} - T_{5;1|2|3,4} \right\rangle = 0 \tag{116}$$

$$\left\langle k_1^m T^m_{1,2,3|4,5} + T_{1;4|5|2,3} + T_{1;5|4|2,3} + T_{12,3|4,5} + T_{13,2|4,5} \right\rangle = 0$$

which can be verified to be true using the identities in (102). Alternatively, their validity also follows from the fact that these are BRST-closed linear combinations of local building blocks and that the five-point local cohomology is empty.

利用 (102) 中的恒等式可验证该式成立。换言之，这些关系的正确性也可由以下事实推出: 它们是局部构造块的 BRST 闭线性组合，且五点局部上同调为空。

Genus Three

亏格三

Four points The chiral correlator for four external massless states was determined in [46] up to terms that have no singularities on the worldsheet (We note a recent [87] conjecture for the full bosonic correlator obtained from matching its field-theory limit with the $\mathcal{N} = 8$ integrand in a BCJ parameterization.). It can be written as

四点四个外质量零态的手征关联函数已在文献 [46] 中确定，仅相差世界面上无奇异的项 (我们注意到近期文献 [87] 提出了一个完全玻色关联函数的猜想，该猜想通过将其场论极限与 BCJ 参数化中的 $\mathcal{N} = 8$ 被积函数匹配得到)。它可以写为

$$\mathcal{K}_4(\ell) = T^m_{1,4|2|3} \ell^I_m w^I_1 \Delta_{234} + T^m_{2,4|1|3} \ell^I_m w^I_2 \Delta_{134} + T^m_{3,4|1|2} \ell^I_m w^I_3 \Delta_{124} \tag{117}$$

$$+ T_{12|3|4} \Delta_{234} \eta_{12} + T_{13|2|4} \Delta_{324} \eta_{13} + T_{14|2|3} \Delta_{423} \eta_{14}$$

$$+ T_{23|1|4} \Delta_{314} \eta_{23} + T_{24|1|3} \Delta_{413} \eta_{24} + T_{34|1|2} \Delta_{412} \eta_{34}$$

where $\Delta_{ijk} = \varepsilon^{IJK} \omega_I(z_i) \omega_J(z_j) \omega_K(z_k)$ for $I, J, K = 1, 2, 3$ and η_{ij} is a worldsheet function depending on the genus-three prime form $E(z_i, z_j)$

其中对应 $I, J, K = 1, 2, 3$ 和 η_{ij} 的 $\Delta_{ijk} = \varepsilon^{IJK} \omega_I(z_i) \omega_J(z_j) \omega_K(z_k)$ 是一个依赖于亏格三素形式 $E(z_i, z_j)$ 的世界面函数

$$\eta_{ij} = \frac{\partial}{\partial z_i} \ln E\left(z_i, z_j\right). \tag{118}$$

The building blocks above depend on non-minimal pure spinor fields. The vectorial building block is constructed as follows:

上述构造块依赖于非极小纯旋量场。矢量构造块的构造如下:

$$T^m_{1,2|3|4} = L^m_{1342} + L^m_{2341} + \frac{5}{2} S^m_{1234} \tag{119}$$

where $S^m_{1234} = S^{(1)m}_{1234} + S^{(2)m}_{1234} - S^{(2)m}_{1243}$ and

其中 $S^m_{1234} = S^{(1)m}_{1234} + S^{(2)m}_{1234} - S^{(2)m}_{1243}$ 和

$$S^{(1)m}_{1234} = 2\left(\bar\lambda\gamma^m\gamma^{a_1}\lambda\right)\left(\bar\lambda\gamma_{m_1n_1p_1}r\right)\left(\bar\lambda\gamma_{m_2n_2p_2}r\right)\left(\bar\lambda\gamma_{m_3n_3p_3}r\right)\left(\bar\lambda\gamma_{m_4n_4p_4}r\right)\left(\bar\lambda\gamma_{m_5n_5p_5}r\right)$$

$$\times\left(\lambda\gamma^{a_2m_1n_1p_1m_3}\lambda\right)\left(\lambda\gamma^{a_3m_2n_2p_2m_5}\lambda\right)\left(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda\right)$$

$$\times\left(W^1\gamma^{a_1a_2a_3}W^2\right)\left(\lambda\gamma^{p_3}W^3\right)\left(\lambda\gamma^{p_5}W^4\right)$$

(120)

$$S^{(2)m}_{1234} = 96\left(\bar\lambda\gamma^m\gamma^{m_3}\lambda\right)\left(\bar\lambda\gamma_{m_1n_1p_1}r\right)\left(\bar\lambda\gamma_{m_2n_2p_2}r\right)\left(\bar\lambda\gamma_{m_3n_3p_3}r\right)\left(\bar\lambda\gamma_{m_4n_4p_4}r\right)$$

$$\left(\bar\lambda\gamma_{m_5n_5p_5}r\right)$$

$$\times\left(\lambda\gamma^{m_1m_2n_2p_2m_5}\lambda\right)\left(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda\right)$$

$$\times\left(\lambda\gamma^{n_1}W^1\right)\left(\lambda\gamma^{p_1}W^2\right)\left(\lambda\gamma^{p_3}W^3\right)\left(\lambda\gamma^{p_5}W^4\right)$$

$$L^m_{ijkl} = \left(\bar\lambda\gamma^{abc}r\right)\left(\bar\lambda\gamma^{def}r\right)\left(\bar\lambda\gamma^{ghi}r\right)\left(\bar\lambda\gamma^{mnp}r\right)\left(\bar\lambda\gamma^{qrs}r\right)\left(\bar\lambda\gamma^{tuv}r\right)$$

$$\times\left(\lambda\gamma^{adefm}\lambda\right)\left(\lambda\gamma^{bghit}\lambda\right)\left(\lambda\gamma^{uqrsn}\lambda\right)\left(\lambda\gamma^cW_i\right)\left(\lambda\gamma^pW_j\right)\left(\lambda\gamma^vW_k\right)A^m_l.$$

The scalar building block is given by

标量构造块由下式给出

$$T_{ij|k|l} = \left(\bar\lambda\gamma^{abc}r\right)\left(\bar\lambda\gamma^{def}r\right)\left(\bar\lambda\gamma^{ghi}r\right)\left(\bar\lambda\gamma^{mnp}r\right)\left(\bar\lambda\gamma^{qrs}r\right)\left(\bar\lambda\gamma^{tuv}r\right) \tag{121}$$

$$\times\left(\lambda\gamma^{\text{ adefm }}\lambda\right)\left(\lambda\gamma^{\text{ bghit }}\lambda\right)\left(\lambda\gamma^{\text{ uqrsn }}\lambda\right)\left(\lambda\gamma^cW_{ij}\right)\left(\lambda\gamma^pW_k\right)\left(\lambda\gamma^vW_l\right).$$

The presence of the non-minimal fields $\overline{\lambda}_\alpha, r_\beta$ in these building blocks leads to technical challenges that do not exist when dealing with "minimal" pure spinor superspace expressions. The r_β fields can be straightforwardly converted into superspace derivatives D_β , but the handling of $\overline{\lambda}_\alpha$ is not so immediate. But luckily, as proven in the appendix of [46], there exists a procedure to convert an arbitrary non-minimal pure spinor superspace expression containing $\overline{\lambda}^n \lambda^{n+3}$ pure spinors with $n \geq 1$ into an expression in which the $\overline{\lambda}_\alpha$ are contracted with λ^α yielding "minimal" pure spinor superspace expressions with $\left(\lambda\overline{\lambda}\right)^n \lambda^3$. As the $\left(\lambda\overline{\lambda}\right)^n$ factor only affects the normalization of the zero-mode integration, one can consider these non-minimal pure spinor superspace expressions more or less in the same footing as their minimal counterparts. It is worth mentioning that there is a proposal for these building blocks directly in minimal pure spinor superspace using higher-mass SYM superfields as [71]

这些构造块中存在非极小场 $\overline{\lambda}_\alpha, r_\beta$ ，带来了处理“极小”纯旋量超空间表达式时不存在的技术困难。r_β 场可以直接转换为超空间导数 D_β ，但处理 $\overline{\lambda}_\alpha$ 没那么简单。不过幸运的是，正如文献 [46] 附录中证明的，存在一套方法，可以将任意包含 $\overline{\lambda}^n \lambda^{n+3}$ 纯旋量 (带有 $n \geq 1$) 的非极小纯旋量超空间表达式转换为 $\overline{\lambda}_\alpha$ 与 λ^α 缩并的表达式，最终得到带有 $\left(\lambda\overline{\lambda}\right)^n \lambda^3$ 的极小纯旋量超空间表达式。由于 $\left(\lambda\overline{\lambda}\right)^n$ 因子仅影响零模积分的归一化，这些非极小纯旋量超空间表达式基本可以和极小情形同等处理。值得一提的是，正如文献 [71] 所述，目前已有方案直接在极小纯旋量超空间中利用高质量超杨-米尔斯超场给出这些构造块

$$T_{12,3,4} \equiv \left\langle (\lambda\gamma_m W_{12}^n)\left(\lambda\gamma_n W_{[3}^p\right)\left(\lambda\gamma_p W_{4]}^m\right)\right\rangle$$

$$T_{1234}^m \equiv \left\langle A_{(1}^m T_{2),3,4} + \left(\lambda\gamma^m W_{(1}\right) L_{2),3,4}\right\rangle \tag{122}$$

$$L_{2,3,4} \equiv \frac{1}{3}\left(\lambda\gamma^n W_{[2}^q\right)\left(\lambda\gamma^q W_3^p\right) F_{4]}^{np}\rangle,$$

where $W_P^{m\alpha}$ represents a local superfield of higher-mass dimension as defined in [71]; when $P = i$ is a single letter, it reduces to $W_i^{m\alpha} = k_i^m W_i^\alpha$, but when P is a word, there are nontrivial contact-term corrections. The component expansion of these building blocks is not exactly the same as their non-minimal counterparts, but they yield the same $D^6 R^4$ components as discussed below.

其中 $W_P^{m\alpha}$ 代表文献 [71] 中定义的高质量维度局域超场；当 $P = i$ 为单个字母时，它约化为 $W_i^{m\alpha} = k_i^m W_i^\alpha$ ，但当 P 为多个字母时，存在非平凡接触项修正。这些构造块的分量展开与非极小对应形式并不完全相同，但它们会得到和下文讨论相同的 $D^6 R^4$ 分量。

After approximating the Koba-Nielsen factor to one in the low-energy limit and integrating over the volume of moduli space, the holomorphic square of the integrand

在低能极限下将木下-尼尔森因子近似为 1 并对模空间体积积分后，被积函数的全纯平方

$$\frac{\left|T_{12,3,4}\right|^2}{s_{12}} + \left|T_{1234}^m\right|^2 + (1,2 \mid 1,2,3,4) \tag{123}$$

is proportional to the $D^6 R^4$ interaction of type II when expanded in bosonic components, regardless of the minimal vs non-minimal representations of the building blocks. One can show that the low-energy contribution (123) is BRST closed, but not the chiral correlator (117). A BRST-closed and single-valued chiral correlator to all orders in α' has since been found [76].

按玻色分量展开后正比于 II 型的 D^6R^4 相互作用，与构造块是极小还是非极小表示无关。可以证明，低能贡献 (123) 是 BRST 闭的，但手征关联函数 (117) 不是。此后人们已经找到了对 α' 所有阶都成立的 BRST 闭且单值的手征关联函数 [76]。

It is worth noting that the computations of [46] were done keeping track of the absolute normalizations coming from the pure spinor prescription with the integration formulas from [47]. As will be reviewed below, these calculations matched the predictions to the D^6R^4 type IIB interaction arising from the S-duality considerations of Green and Vanhove [54].

值得注意的是，文献 [46] 的计算全程保留了来自纯旋量方案结合文献 [47] 积分公式的绝对归一化。正如下文将要回顾的，这些计算与 Green 和 Vanhove 基于 S 对偶考虑得到的 D^6R^4 IIB 型相互作用的预言一致 [54]。

Verifying S-Duality Conjectures

验证 S 对偶猜想

The scattering amplitudes computed with pure spinor formalism have provided an independent check on the S-duality predictions of type IIB interactions.

用纯自旋形式学计算出的散射振幅，已经对 IIB 型相互作用的 S 对偶预言给出了独立检验。

S-Duality and Four-Point Amplitudes

S 对偶与四点振幅

On the one hand, the $SL(2,\mathbb{Z})$ -duality prediction for the perturbative four-graviton type IIB effective action in the string frame is given by [51-54]

一方面，弦框架下 IIB 型微扰四引力子有效作用量的 $SL(2,\mathbb{Z})$ 对偶预言由文献 [51-54] 给出

$$S_{\mathrm{IIB}}^{4\mathrm{pt}} = \int d^{10}x\sqrt{-g}\Big[R^4\left(2\zeta_3 e^{-2\phi}+4\zeta_2\right)+D^4R^4\left(2\zeta_5 e^{-2\phi}+\frac{8}{3}\zeta_4 e^{2\phi}\right) \tag{124}$$

$$+D^6R^4\left(4\zeta_3^2 e^{-2\phi}+8\zeta_2\zeta_3+\frac{48}{5}\zeta_2^2 e^{2\phi}+\frac{8}{9}\zeta_6 e^{4\phi}\right)+\cdots\Big],$$

where the shorthands R^4, D^4R^4 , and D^6R^4 denote contractions of covariant derivatives D and Riemann curvature tensors R whose precise structure does not affect the analysis. Factors of $e^{(2g-2)\phi}$ are associated with the genus- g order in string perturbation theory. The key idea of the S-duality analysis was to associate the coefficients of the R^4 interaction with the zero modes of non-holomorphic Eisenstein series $E_{3/2}\left(\Phi,\overline{\Phi}\right)$ and those of D^4R^4 with $E_{5/2}\left(\Phi,\overline{\Phi}\right)$, where

其中简写记号 R^4, D^4R^4 和 D^6R^4 表示协变导数 D 与黎曼曲率张量 R 的缩并，它们的具体结构不影响本文分析。$e^{(2g-2)\phi}$ 因子对应弦微扰论中亏格 g 阶的贡献。S 对偶分析的核心思路是: 将 R^4 相互作用的系数与非全纯艾森斯坦级数 $E_{3/2}\left(\Phi,\overline{\Phi}\right)$ 的零模联系起来，将 D^4R^4 相互作用的系数与 $E_{5/2}\left(\Phi,\overline{\Phi}\right)$ 联系起来，其中

$$E_{3/2}\left(\Phi,\overline{\Phi}\right)=2\zeta_3e^{-3\phi/2}+4\zeta_2e^{\phi/2}+\cdots \tag{125}$$

$$E_{5/2}\left(\Phi,\overline{\Phi}\right)=2\zeta_5e^{-5\phi/2}+\frac{8}{3}\zeta_4e^{3\phi/2}+\cdots$$

where Φ depends on the complex axio-dilaton field $\Phi=C_0+ie^{-\phi}$.

其中 Φ 依赖于复轴子- dilaton 场 $\Phi=C_0+ie^{-\phi}$ 。

On the other hand, the α' expansion of perturbative string scattering amplitude calculations performed with the non-minimal pure spinor formalism with the absolute normalization techniques from [45-47] for the four-point massless closed-string states is given by

另一方面，针对四点无质量闭弦态，采用非最小纯自旋形式主义结合文献 [45-47] 的绝对归一化技术得到的微扰弦散射振幅的 α' 展开式为

$$M_4^{(0)}=(2\pi)^{10}\delta^{10}(k)\left(\frac{\alpha'}{2}\right)^3\kappa^4e^{-2\lambda}2\pi K\widetilde{K}\left(\frac{3}{\sigma_3}+2\zeta_3+\zeta_5\sigma_2+\frac{2}{3}\zeta_3^2\sigma_3+\cdots\right)$$

$$M_4^{(1)}=(2\pi)^{10}\delta^{10}(k)\left(\frac{\alpha'}{2}\right)^3\kappa^4\left(\frac{1}{2^4\cdot 3\pi}\right)K\widetilde{K}\left(1+\frac{\zeta_3}{3}\sigma_3+\cdots\right)$$

$$M_4^{(2)}=(2\pi)^{10}\delta^{10}(k)\left(\frac{\alpha'}{2}\right)^3\kappa^4e^{2\lambda}\left(\frac{1}{2^{10}\cdot 3^3\cdot 5\pi^3}\right)K\widetilde{K}\left(\sigma_2+3\sigma_3+\cdots\right)$$

$$M_4^{(3)}=(2\pi)^{10}\delta^{10}(k)\left(\frac{\alpha'}{2}\right)^3\kappa^4e^{4\lambda}\left(\frac{1}{2^{15}\cdot 3^6\cdot 5\cdot 7\pi^5}\right)K\widetilde{K}\left(\sigma_3+\cdots\right) \tag{126}$$

where

其中

$$\sigma_n=\left(\frac{\alpha'}{2}\right)^n\left(s_{12}^n+s_{13}^n+s_{14}^n\right) \tag{127}$$

are dimensionless symmetric polynomials of Mandelstam invariants $s_{ij}=\left(k_i\cdot k_j\right)$, $e^{(2g-2)\lambda}$ is the string genus- g coupling constant, K is the supersymmetric kinematic factor (For bosonic external states, note that $-2^3K^{\text{ here}}=K^{0503}$ from [43] and that $K^{\text{ here}}\widetilde{K}^{\text{ here}}=\mathcal{K}_4^{(0)}$ from [48]. In addition, the amplitudes in (126) were computed using the tree-level normalization convention encoded by $R^2=\pi^5/2^5$ used in [48] where R is a normalization parameter appearing in the zero-mode measures $[dr]$ and $\left[ds^I\right]$. In [46] the normalization $R^2=\sqrt{2}/\left(2^{16}\pi\right)$ was chosen, such that the genus g amplitudes A_g^{1308} of [46] are related by $x^{1-g}A_g^{1308}=M_g^{1504}$ to the amplitudes M_g^{1504} of [48] with $x=\sqrt{2}2^{10}\pi^6$ after considering that $K^{1308}\bar{K}^{1308}=2^6\mathcal{K}_4^{(0)}$.)

是曼德尔施塔姆不变量的无量纲对称多项式，$s_{ij}=\left(k_i\cdot k_j\right)$、$e^{(2g-2)\lambda}$ 是弦亏格 g 耦合常数，K 是超对称运动学因子 (对于玻色外态，注意 $-2^3K^{\text{ here}}=K^{0503}$ 来自文献 [43]，$K^{\text{ here}}\widetilde{K}^{\text{ here}}=\mathcal{K}_4^{(0)}$ 来自文献 [48]。此外，(126) 式中的振幅是按照文献 [48] 所用 $R^2=\pi^5/2^5$ 编码的树级归一化约定计算的，其中 R 是出现在零模测度 $[dr]$ 和 $\left[ds^I\right]$ 中的归一化参数。文献 [46] 选取了归一化条件 $R^2=\sqrt{2}/\left(2^{16}\pi\right)$，因此在考虑 $K^{1308}\bar{K}^{1308}=2^6\mathcal{K}_4^{(0)}$ 后，文献 [46] 的亏格 g 振幅 A_g^{1308} 与文献 [48] 的振幅 M_g^{1504} 满足关系 $x^{1-g}A_g^{1308}=M_g^{1504}$，此时 $x=\sqrt{2}2^{10}\pi^6$。)

$$K=s_{12}s_{23}A^{\text{SYM}}(1,2,3,4)\tag{128}$$

and κ is the normalization of the vertex operators fixed to $\kappa^2=e^{2\lambda}\pi/\alpha'^2$ by unitarity [48].

且 κ 是顶点算符的归一化，由幺正性固定为 $\kappa^2=e^{2\lambda}\pi/\alpha'^2$ [48]。

It is easy to see the one-to-one correspondence of the genus- and α' -orders in the amplitudes (126) with the curvature couplings in the action (124)

容易看出，振幅 (126) 中亏格阶与 α' 阶，和作用量 (124) 中的曲率耦合存在一一对应关系

$$e^{(2g-2)\phi}D^{2k}R^4\leftrightarrow e^{(2g-2)\lambda}K\widetilde{K}\sigma_k.\tag{129}$$

Therefore, matching the ratio (genus one)/(genus zero) of the R^4 interactions in the effective action with the corresponding ratio of the amplitudes

因此，将有效作用量中 R^4 相互作用的 (亏格一)/(亏格零) 比值与振幅的对应比值匹配

$$\frac{4\zeta_2R^4}{2\zeta_3e^{-2\phi}R^4}=\frac{e^{2\phi}\pi^2}{3\zeta_3},\ \frac{\left(K\widetilde{K}\right)/\left(2^43\pi\right)}{\left(K\widetilde{K}\right)4\pi\zeta_3e^{-2\lambda}}=\frac{e^{2\lambda}}{2^63\pi^2\zeta_3}\tag{130}$$

relates the coupling constants e^ϕ and e^λ ,

可以得到耦合常数 e^ϕ 和 e^λ 之间的关系，

$$e^{2\lambda}=2^6\pi^4e^{2\phi}.\tag{131}$$

D^4R^4 interaction at genus two One can now compare the predicted D^4R^4 interaction terms from the type IIB effective action (124) with the first principle string calculations. Taking the genus-two/genus-zero ratio of the $K\widetilde{K}\sigma_2$ term from the amplitudes gives:

亏格二下的 D^4R^4 相互作用现在我们可以将 IIB 型有效作用量 (124) 给出的 D^4R^4 相互作用项预言与第一性原理弦计算结果进行比较。对振幅中 $K\widetilde{K}\sigma_2$ 项取亏格二与亏格零的比值可得:

$$\frac{e^{4\lambda}}{2^{11}3^35\pi^4\zeta_5}=\frac{2\pi^4e^{4\phi}}{3^35\zeta_5}\tag{132}$$

where we used (131) in the right-hand side. This is the same ratio of the D^4R^4 terms in (124) at the corresponding loop order: $8\zeta_4 e^{4\phi}/(6\zeta_5) = 2\pi^4 e^{4\phi}/\left(3^3 5\zeta_5\right)$ as $\zeta_4 = \pi^4/\left(2\cdot 3^2\cdot 5\right)$.

其中我们在右侧使用了 (131) 式。这与对应圈阶下 (124) 式中 D^4R^4 项的比值完全一致: $8\zeta_4 e^{4\phi}/(6\zeta_5) = 2\pi^4 e^{4\phi}/\left(3^3 5\zeta_5\right)$ 即 $\zeta_4 = \pi^4/\left(2\cdot 3^2\cdot 5\right)$.

D^6R^4 interaction at genus three Similarly, the ratio (genus three)/(genus one) correction $K\widetilde{K}\sigma_3$ matches perfectly with the S-duality result in (124). The ratio of the amplitudes is given by $e^{4\lambda}/\left(2^{11}\cdot 3^4\cdot 5\cdot 7\pi^4\zeta_3\right)$, while in the effective action, it is given by $\zeta_6 e^{4\phi}/(9\zeta_2\zeta_3)$, and these two numbers match after using the conversion (131) and $\zeta_6 = \pi^6/\left(3^3\cdot 5\cdot 7\right)$.

亏格三下的 D^6R^4 相互作用类似地, 修正项 $K\widetilde{K}\sigma_3$ 的 (亏格三)/(亏格一) 比值与 (124) 式中的 S 对偶结果完全匹配。振幅给出的比值为 $e^{4\lambda}/\left(2^{11}\cdot 3^4\cdot 5\cdot 7\pi^4\zeta_3\right)$，而有效作用量给出的比值为 $\zeta_6 e^{4\phi}/(9\zeta_2\zeta_3)$ ，代入变换式 (131) 与 $\zeta_6 = \pi^6/\left(3^3\cdot 5\cdot 7\right)$ 后，二者数值一致。

D^6R^4 interaction at genus two The coefficient of $K\widetilde{K}\sigma_3$ at genus-two was computed in [44] and allowed the comparison between the string scattering amplitude result at genus two with the S-duality prediction in the action (124). The (genus two)/(genus one) ratio of the correction $K\widetilde{K}\sigma_3$ is given by $e^{2\lambda}/\left(2^6 5\pi^2\zeta_3\right)$ which matches the S-duality (genus two)/(genus one) ratio of the D^6R^4 interaction, given by $6\zeta_2 e^{2\phi}/(5\zeta_3)$ after using the conversion (131) and $\zeta_2 = \pi^2/6$.

亏格二下的 D^6R^4 相互作用文献 [44] 计算得到了亏格二下 $K\widetilde{K}\sigma_3$ 的系数，由此可以将亏格二弦散射振幅结果与作用量 (124) 中的 S 对偶预言进行比较。修正项 $K\widetilde{K}\sigma_3$ 的 (亏格二)/(亏格一) 比值为 $e^{2\lambda}/\left(2^6 5\pi^2\zeta_3\right)$，代入变换式 (131) 与 $\zeta_2 = \pi^2/6$ 后，它与 D^6R^4 相互作用的 S 对偶 (亏格二)/(亏格一) 比值 $6\zeta_2 e^{2\phi}/(5\zeta_3)$ 一致。

S-Duality and Five-Point Amplitudes

S 对偶与五点振幅

The pure spinor formalism also allowed to check the S-duality proposals for five graviton interactions as well as four gravitons and one dilaton. For five gravitons the S-duality effective action contains the same ratios appearing in the four-graviton action (124); the extension is straightforward with four-curvature corrections $D^{2k}R^4$ followed by a tail of operators $D^{2(k-p)}R^{4+p}$, although there might be novel $D^{2k}R^{\geq 5}$ couplings without a four-field counterpart such as the D^6R^5 interaction at genus one [55]. These S-duality tails such as $\left(D^4R^4 + D^2R^5\right)$ are confirmed by the data of the genus- g amplitudes $M_5^{(g)}$ at five points (To avoid cluttering, we omit the universal factor of $(2\pi)^{10}\delta^{10}(k)$ from the right-hand side of (133).) [48]

纯自旋形式化也可用于检验五引力子相互作用，以及四引力子加一 dilaton 相互作用的 S 对偶猜想。对于五引力子, S 对偶有效作用量包含四引力子作用量 (124) 中已出现的相同比值; 其扩展十分直接, 即四曲率修正 $D^{2k}R^4$ 后接算符尾项 $D^{2(k-p)}R^{4+p}$ ，不过也可能存在不存在四场对应项的新 $D^{2k}R^{\geq 5}$ 耦合，例如亏格 1 处的 D^6R^5 相互作用 [55]。这些 S 对偶尾项 (如 $\left(D^4R^4 + D^2R^5\right)$) 已由五点亏格- g 振幅 $M_5^{(g)}$ 的数据证实 (为避免杂乱，我们从 (133) 的右侧省略了 $(2\pi)^{10}\delta^{10}(k)$ 的普适因子。)[48]

$$M_5^{(0)} = \left(\frac{\alpha'}{2}\right)\kappa^5 e^{-2\lambda}(2\pi)^2\mathcal{K}_5^{(0)} \tag{133}$$

$$M_5^{(1)}\Big|_{\rm IIB}^{\alpha'^4} = \left(\frac{\alpha'}{2}\right)\frac{\kappa^5}{2^4 3}\mathcal{K}_5^{(0)}\Big|_{\zeta_3} \times \begin{cases} 1: & \text{five gravitons} \\ -\frac{1}{3}: & \text{four gravitons, one dilaton} \end{cases}$$

$$M_5^{(2)}\Big|_{\rm IIB}^{\alpha'^6} = \left(\frac{\alpha'}{2}\right)\frac{\kappa^5 e^{2\lambda}}{2^9 3^3 5\pi^2}\mathcal{K}_5^{(0)}\Big|_{\zeta_5} \times \begin{cases} 1: & \text{five gravitons} \\ -\frac{3}{5}: & \text{four gravitons, one dilaton} \end{cases}$$

where the tree-level factor $\mathcal{K}_5^{(0)}$ is given by

其中树图因子 $\mathcal{K}_5^{(0)}$ 由下式给出

$$\mathcal{K}_5^{(0)} = \widetilde{A}_{54}^T \cdot S_0 \cdot \left[1 + 2\zeta_3\left(\frac{\alpha'}{2}\right)^3 M_3 + 2\zeta_5\left(\frac{\alpha'}{2}\right)^5 M_5 + 2\zeta_3^2\left(\frac{\alpha'}{2}\right)^6 M_3^2 + \mathcal{O}\left(\alpha'^7\right)\right] \cdot A_{45},$$

(134)

where $\widetilde{A}_{54}^T$ and A_{45} are two-component vectors of SYM tree amplitudes

其中 $\widetilde{A}_{54}^T$ 和 A_{45} 是超杨-米尔斯树振幅的双分量矢量

$$\widetilde{A}_{54} \equiv \begin{pmatrix} \widetilde{A}^{\rm YM}(1,2,3,5,4) \\ \widetilde{A}^{\rm YM}(1,3,2,5,4) \end{pmatrix}, \; A_{45} \equiv \begin{pmatrix} A^{\rm YM}(1,2,3,4,5) \\ A^{\rm YM}(1,3,2,4,5) \end{pmatrix}, \tag{135}$$

S_0 denotes the KLT matrix and the 2×2 matrices M_{2n+1} were introduced in [72].

S_0 表示 KLT 矩阵，矩阵 2×2 和 M_{2n+1} 已在文献 [72] 中引入。

Since the calculations in the pure spinor formalism are supersymmetric and done exploiting pure spinor superspace, the scattering of any state in the graviton supermultiplet can be systematically obtained once the superspace expression is calculated. As can be seen in (133), the ratios of the string amplitudes depend on the R-symmetry charges of the external type IIB states, as trading one graviton for a dilaton gives the additional factors of $-\frac{1}{3}$ or $-\frac{3}{5}$.

由于纯自旋形式化中的计算是超对称的，且利用纯自旋超空间完成，一旦算出超空间表达式，就可以系统地得到引力子超多重态中任意态的散射。从 (133) 中可以看出，弦振幅的比值依赖于 IIB 型弦外部态的 R 对称性荷，因为将一个引力子替换为 dilaton 会得到额外因子 $-\frac{1}{3}$ 或 $-\frac{3}{5}$ 。

These numbers can be explained by the following argument [56]: scattering processes which violate the R-symmetry of type IIB supergravity are associated with operators which transform with modular weight under S-duality; therefore, by modular invariance of the type IIB action, they must be accompanied by modular forms of opposite weights to preserve the modular invariance of the type IIB effective action. These modular forms can be generated as DE_s where D is the modular covariant derivative such that $De^{q\phi} = q \cdot e^{q\phi}$ and E_s a Eisenstein series. For example,

这些数值可以通过下述论证解释 [56]: 破坏 IIB 型超引力 R 对称性的散射过程，对应在 S 对偶下按模权变换的算符；因此，根据 IIB 型作用量的模不变性，这些算符必须伴随相反权的模形式，才能保证 IIB 型有效作用量的模不变性。这些模形式可以构造为 DE_s，其中 D 是模协变导数，满足 $De^{q\phi} = q \cdot e^{q\phi}$，而 E_s 是爱森斯坦级数。例如:

$$DE_{3/2}\left(\Phi, \overline{\Phi}\right) = \left(-\frac{3}{2}\right) 2\zeta_3 e^{-3\phi/2} + \left(\frac{1}{2}\right) 4\zeta_2 e^{\phi/2} + \cdots \tag{136}$$

$$DE_{5/2}\left(\Phi, \overline{\Phi}\right) = \left(-\frac{5}{2}\right) 2\zeta_5 e^{-5\phi/2} + \left(\frac{3}{2}\right) \frac{8}{3} \zeta_4 e^{3\phi/2} + \cdots .$$

Thus, the ratio between tree-level and higher-genus contributions is deformed by $-\frac{1}{3}$ and $-\frac{3}{5}$ in cases of $E_{3/2}$ and $E_{5/2}$, suggesting that the type IIB effective action contains the terms:

因此，在 $E_{3/2}$ 和 $E_{5/2}$ 的情形下，树图与高亏格贡献的比值会被 $-\frac{1}{3}$ 和 $-\frac{3}{5}$ 形变，这说明 IIB 型有效作用量包含如下项:

$$\int d^{10}x\sqrt{-g}\left[\phi R^4\left(-3\zeta_3 e^{-2\phi} + 2\zeta_2\right) + \phi D^4 R^4\left(-5\zeta_5 e^{-2\phi} + 4\zeta_4 e^{2\phi}\right)\right] \tag{137}$$

in the string frame (The term ϕR^4 is multiplied by $e^{-\phi/2}$ and $\phi D^4 R^4$ by $e^{\phi/2}$ in going to the string frame.) and explaining the relative coefficients in the scattering amplitudes (133).

在弦框架下 (转换到弦框架时，项 ϕR^4 会乘以 $e^{-\phi/2}$，$\phi D^4 R^4$ 会乘以 $e^{\phi/2}$)，这解释了散射振幅 (133) 中的相对系数。

Cross-References

交叉引用

- A Lightning Introduction to String Theory

- 弦论闪电简介

Causality and Scattering Amplitudes in Nonlocal Gravity

非局域引力中的因果性与散射振幅

Pure Spinor Formulation of the Superstring and Its Applications

超弦的纯旋量表述及其应用

String Field Theory: A Review

弦场论: 综述

Superembedding Approach to Superstrings and Super-p-Branes

超弦与超 p 膜的超嵌入方法

Acknowledgments CRM is supported by a University Research Fellowship from the Royal Society. NB would like to thank CNPq grant 311434/2020-7 and FAPESP grants 2016/01343-7, 2021/14335-0, 2019/21281-4, and 2019/24277-8 for partial financial support.

致谢 CRM 得到英国皇家学会大学研究奖学金的支持。NB 感谢 CNPq 基金 311434/2020-7 以及 FAPESP 基金 2016/01343-7、2021/14335-0、2019/21281-4 与 2019/24277-8 提供的部分资助。

References

参考文献

1. N. Berkovits, Super Poincare covariant quantization of the superstring. JHEP 04, 018 (2000). https://doi.org/10.1088/112 6708/2000/04/018, [arXiv:hep-th/0001035 [hep-th]]

2. M.B. Green, J.H. Schwarz, Covariant Description of Superstrings. Phys. Lett. B 136, 367-370 (1984). https://doi.org/10.1016/0370-2693(84)92021-5

3. W. Siegel, Classical superstring mechanics. Nucl. Phys. B 263, 93-104 (1986). https://doi.org/10.1016/0550-3213(86)90029-5

4. P.S. Howe, Pure spinors lines in superspace and ten-dimensional supersymmetric theories. Phys. Lett. B 258, 141-144 (1991). https://doi.org/10.1016/0370-2693(91)91221-G

5. P.S. Howe, Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions. Phys. Lett. B 273, 90-94 (1991). https://doi.org/10.1016/0370- 2693(91)90558-8

6. M. Cederwall, B.E.W. Nilsson, D. Tsimpis, Spinorial cohomology and maximally super-symmetric theories. JHEP 02, 009 (2002). https://doi.org/10.1088/1126-6708/2002/02/009, [arXiv:hep-th/0110069 [hep-th]]

7. N. Berkovits, Towards covariant quantization of the supermembrane. JHEP 09, 051 (2002). https://doi.org/10.1088/1126 6708/2002/09/051, [arXiv:hep-th/0201151 [hep-th]]

8. M. Cederwall, D = 11 supergravity with manifest supersymmetry. Mod. Phys. Lett. A 25, 3201-3212 (2010). https://doi.org/10.1142/S0217732310034407, [arXiv:1001.0112 [hep-th]]

9. M. Cederwall, Pure spinors in classical and quantum supergravity. [arXiv:2210.06141 [hep-th]]

10. N. Berkovits, Untwisting the pure spinor formalism to the RNS and twistor string in a flat and AdS $_5 \times S^5$ background. JHEP 06,127 (2016). https://doi.org/10.1007/JHEP06(2016)127, [arXiv:1604.04617 [hep-th]]

11. N. Berkovits, Manifest spacetime supersymmetry and the superstring. JHEP 10, 162 (2021). https://doi.org/10.1007/JH [arXiv:2106.04448 [hep-th]]

12. N. Berkovits, P.S. Howe, Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring. Nucl. Phys. B 635, 75-105 (2002). https://doi.org/10.1016/ S0550-3213(02)00352-8, [arXiv:hep-th/0112160 [hep-th]]

13. N. Berkovits, Quantum consistency of the superstring in AdS(5) x S**5 background. JHEP 03, 041 (2005). https://doi.org/10.1088/1126-6708/2005/03/041, [arXiv:hep-th/0411170 [hep-th]]

14. N. Berkovits, Half-BPS vertex operators of the $AdS_5 \times S^5$ superstring. JHEP 07,084 (2019). https://doi.org/10.1007/JHEP [arXiv:1904.06564 [hep-th]]

15. L.N.S. Martins, Type IIB superstring vertex operator from the -8 picture. Nucl. Phys. B 974, 115638 (2022). https://doi.org/10.1016/j.nuclphysb.2021.115638, [arXiv:1912.06498 [hep-th]]

16. T. Fleury, L.N.S. Martins, AdS5 $\times$ S ^{5} supergravity vertex operators. JHEP 07, 210 (2021). https://doi.org/10.1007/JHEP0 [arXiv:2104.03333 [hep-th]]

17. N. Berkovits, Simplifying and Extending the AdS(5) x S**5 Pure Spinor Formalism. JHEP 09, 051 (2009). https://doi.org/10.1088/1126-6708/2009/09/051, [arXiv:0812.5074 [hep-th]]

18. N. Berkovits, Sketching a proof of the Maldacena conjecture at small radius. JHEP 06, 111 (2019). https://doi.org/10.1007/JHEP06(2019)111, [arXiv:1903.08264 [hep-th]]

19. E. Witten, Perturbative gauge theory as a string theory in twistor space. Commun. Math. Phys. 252, 189-258 (2004). https://doi.org/10.1007/s00220-004-1187-3, [arXiv:hep-th/0312171 [hep-th]]

20. L. Eberhardt, M.R. Gaberdiel, R. Gopakumar, Deriving the AdS $_3$/CFT$_2$ correspondence. JHEP 02, 136 (2020). https://doi.org/10.1007/JHEP02(2020)136, [arXiv:1911.00378 [hep-th]]

21. M.R. Gaberdiel, R. Gopakumar, The worldsheet dual of free super Yang-Mills in 4D. JHEP 11, 129 (2021). https://doi.org/10.1007/JHEP11(2021)129, [arXiv:2105.10496 [hep-th]]

22. N. Berkovits, Pure spinor formalism as an N $=$ 2 topological string. JHEP 10,089 (2005). https://doi.org/10.1088/1126-6708/2005/10/089, [arXiv:hep-th/0509120 [hep-th]]

23. N. Berkovits, Super-Poincare covariant two-loop superstring amplitudes. JHEP 01, 005 (2006). https://doi.org/10.1088/ 6708/2006/01/005, [arXiv:hep-th/0503197 [hep-th]]

24. N. Berkovits, C.R. Mafra, Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms. Phys. Rev. Lett. 96, 011602 (2006). https://doi.org/10.1103/ PhysRevLett.96.011602, [arXiv:hep-th/0509234 [hep-th]]

25. N. Berkovits, Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring. JHEP 09, 047 (2004). https://doi.org/10.1088/1126-6708/2004/09/047, [arXiv:hep-th/0406055 [hep-th]]

26. N. Berkovits, N. Nekrasov, Multiloop superstring amplitudes from non-minimal pure spinor formalism. JHEP 12, 029 (2006). https://doi.org/10.1088/1126-6708/2006/12/029, [arXiv:hep-th/0609012 [hep-th]]

27. N. Berkovits, Explaining Pure Spinor Superspace. [arXiv:hep-th/0612021 [hep-th]]

28. N. Berkovits, C.R. Mafra, Some superstring amplitude computations with the non-minimal pure spinor formalism. JHEP 11, 079 (2006). https://doi.org/10.1088/1126-6708/2006/11/079, [arXiv:hep-th/0607187 [hep-th]]

29. Z. Bern, J.J.M. Carrasco, H. Johansson, New relations for gauge-theory amplitudes. Phys. Rev. D 78, 085011 (2008). https://doi.org/10.1103/PhysRevD.78.085011, [arXiv:0805.3993 [hep-ph]]

30. Z. Bern, L.J. Dixon, D.C. Dunbar, D.A. Kosower, One loop n point gauge theory amplitudes, unitarity and collinear limits. Nucl. Phys. B 425, 217-260 (1994). https://doi.org/10.1016/ 0550-3213(94)90179-1, [arXiv:hep-ph/9403226 [hep-ph]]

31. D. Blessenohl, H. Laue, Generalized Jacobi identities. Note Mat. 8(1), 111-121 (1988). https:// doi.org/10.1285/i15900932v8n1p111

32. E. Bridges, C.R. Mafra, Algorithmic construction of SYM multiparticle superfields in the BCJ gauge. JHEP 10, 022 (2019). https://doi.org/10.1007/JHEP10(2019)022, [arXiv:1906.12252 [hep-th]]

33. E. Bridges, C.R. Mafra, Local BCJ numerators for ten-dimensional SYM at one loop. JHEP 07, 031 (2021). https://doi.org/10.1007/JHEP07(2021)031, [arXiv:2102.12943 [hep-th]]

34. J. Broedel, C.R. Mafra, N. Matthes, O. Schlotterer, Elliptic multiple zeta values and one-loop superstring amplitudes. JHEP 07, 112 (2015). https://doi.org/10.1007/JHEP07(2015)112, [arXiv:1412.5535 [hep-th]]

35. J. Broedel, O. Schlotterer, S. Stieberger, T. Terasoma, All order α' -expansion of superstring trees from

the Drinfeld associator. Phys. Rev. D 89(6), 066014 (2014). https://doi.org/10.1103/ PhysRevD.89.066014, [arXiv:1304.7304 [hep-th]]

36. F. Brown, A. Levin, Multiple elliptic polylogarithms (2011). [arXiv:1110.6917 [math.NT]]

37. D. Friedan, E.J. Martinec, S.H. Shenker, Conformal Invariance, Supersymmetry and String Theory. Nucl. Phys. B 271, 93-165 (1986). https://doi.org/10.1016/0550-3213(86)90356-1

38. E. D'Hoker, D.H. Phong, The geometry of string perturbation theory. Rev. Mod. Phys. 60, 917 (1988). https://doi.org/10.1103/RevModPhys.60.917

39. E. D'Hoker, D.H. Phong, Conformal scalar fields and chiral splitting on super Riemann surfaces. Commun. Math. Phys. 125, 469 (1989). https://doi.org/10.1007/BF01218413

40. E. D'Hoker, D.H. Phong, Two-loop superstrings VI: non-renormalization theorems and the 4- point function. Nucl. Phys. B 715, 3-90 (2005). https://doi.org/10.1016/j.nuclphysb.2005.02.043, [arXiv:hep-th/0501197 [hep-th]]

41. E. D'Hoker, C. R. Mafra, B. Pioline, O. Schlotterer, Two-loop superstring five-point amplitudes. Part I. Construction via chiral splitting and pure spinors. JHEP 08, 135 (2020). https:// doi.org/10.1007/JHEP08(2020)135, [arXiv:2006.05270 [hep-th]]

42. E. D'Hoker, C.R. Mafra, B. Pioline, O. Schlotterer, Two-loop superstring five-point amplitudes. Part II. Low energy expansion and S-duality. JHEP 02, 139 (2021). https://doi.org/10.1007/ JHEP02(2021)139, [arXiv:2008.08687 [hep-th]]

43. E. D'Hoker, M. Gutperle, D.H. Phong, Two-loop superstrings and S-duality. Nucl. Phys. B 722, 81-118 (2005). https://doi.org/10.1016/j.nuclphysb.2005.06.010, [arXiv:hep-th/0503180 [hep-th]]

44. E. D' Hoker, M.B. Green, B. Pioline, R. Russo, Matching the D^6R^4 interaction at two-loops. JHEP 01, 031 (2015). https://doi.org/10.1007/JHEP01(2015)031, [arXiv:1405.6226 [hep-th]]

45. H. Gomez, C.R. Mafra, The overall coefficient of the two-loop superstring amplitude using pure spinors. JHEP 05, 017 (2010). https://doi.org/10.1007/JHEP05(2010)017, [arXiv:1003.0678 [hep-th]]

46. H. Gomez, C.R. Mafra, The closed-string 3-loop amplitude and S-duality. JHEP 10, 217 (2013). https://doi.org/10.1007/JHEP10(2013)217, [arXiv:1308.6567 [hep-th]]

47. H. Gomez, One-loop superstring amplitude from integrals on pure spinors space. JHEP 12, 034 (2009). https://doi.org/10.1088/1126-6708/2009/12/034, [arXiv:0910.3405 [hep-th]]

48. H. Gomez, C.R. Mafra, O. Schlotterer, Two-loop superstring five-point amplitude and S-duality. Phys. Rev. D 93(4), 045030 (2016). https://doi.org/10.1103/PhysRevD.93.045030, [arXiv:1504.02759 [hep-th]]

49. M.B. Green, J.H. Schwarz, The hexagon gauge anomaly in type I superstring theory. Nucl. Phys. B 255, 93 (1985)

50. M.B. Green, J.H. Schwarz, Anomaly cancellation in supersymmetric D = 10 gauge theory and superstring theory. Phys. Lett. B 149, 117 (1984)

51. M.B. Green, M. Gutperle, Effects of D instantons. Nucl. Phys. B 498, 195-227 (1997). https:// doi.org/10.1016/S0550-3213(97)00269-1, [arXiv:hep-th/9701093 [hep-th]]

52. M.B. Green, M. Gutperle, P. Vanhove, One loop in eleven-dimensions. Phys. Lett. B 409, 177- 184 (1997). https://doi.org/10.1016/S0370-2693(97)00931-3, [arXiv:hep-th/9706175 [hep-th]]

53. M.B. Green, H.H. Kwon, P. Vanhove, Two loops in eleven-dimensions. Phys. Rev. D 61, 104010 (2000). https://doi.org/10.1103/PhysRevD.61.104010, [arXiv:hep-th/9910055 [hep-th]]

54. M.B. Green, P. Vanhove, Duality and higher derivative terms in M theory. JHEP 01, 093 (2006). https://doi.org/10.1088/1126-6708/2006/01/093, [arXiv:hep-th/0510027 [hep-th]]

55. M.B. Green, C.R. Mafra, O. Schlotterer, Multiparticle one-loop amplitudes and S-duality in closed superstring theory. JHEP 10, 188 (2013). https://doi.org/10.1007/JHEP10(2013)188, [arXiv:1307.3534 [hep-th]]

56. M.B. Green, M. Gutperle, H.H. Kwon, Sixteen fermion and related terms in M theory on T**2. Phys. Lett. B 421, 149-161 (1998). https://doi.org/10.1016/S0370-2693(97)01551-7, [arXiv:hep-th/9710151 [hep-th]]

57. M.B. Green, J.H. Schwarz, L. Brink, N = 4 Yang-Mills and N = 8 supergravity as limits of string theories. Nucl. Phys. B 198, 474-492 (1982). https://doi.org/10.1016/0550-3213(82)90336-4

58. M.B. Green, J.H. Schwarz, Supersymmetrical dual string theory: III. Loops and renormalization. Nucl. Phys. B 198, 441-460 (1982). https://doi.org/10.1016/0550-3213(82)90334-0

59. C.R. Mafra, Towards field theory amplitudes from the cohomology of pure spinor superspace. JHEP 11, 096 (2010). https://doi.org/10.1007/JHEP11(2010)096, [arXiv:1007.3639 [hep-th]]

60. C.R. Mafra, O. Schlotterer, S. Stieberger, D. Tsimpis, A recursive method for SYM n-point tree amplitudes. Phys. Rev. D 83, 126012 (2011). https://doi.org/10.1103/PhysRevD.83.126012, [arXiv:1012.3981 [hep-th]]

61. C.R. Mafra, O. Schlotterer, S. Stieberger, Complete N-point superstring disk amplitude I. Pure spinor computation. Nucl. Phys. B 873, 419-460 (2013). https://doi.org/10.1016/j.nuclphysb.2013.04.023, [arXiv:1106.2645 [hep-th]]

62. C.R. Mafra, O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part I. Pure spinors and superfield kinematics. JHEP 08, 090 (2019). https://doi.org/10.1007/ JHEP08(2019)090, [arXiv:1812.10969 [hep-th]]

63. C.R. Mafra, O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part II. Worldsheet functions and their duality to kinematics. JHEP 08, 091 (2019). https://doi.org/ 10.1007/JHEP08(2019)091, [arXiv:1812.10970 [hep-th]]

64. C.R. Mafra, O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part III. One-loop correlators and their double-copy structure. JHEP 08, 092 (2019). https://doi.org/10.1007/ JHEP08(2019)092, [arXiv:1812.10971 [hep-th]]

65. C.R. Mafra, O. Schlotterer, Towards one-loop SYM amplitudes from the pure spinor BRST cohomology. Fortsch. Phys. 63(2), 105-131 (2015). https://doi.org/10.1002/prop.201400076, [arXiv:1410.0668 [hep-th]]

66. C.R. Mafra, O. Schlotterer, Multiparticle SYM equations of motion and pure spinor BRST blocks. JHEP 07, 153 (2014). https://doi.org/10.1007/JHEP07(2014)153, [arXiv:1404.4986 [hep-th]]

67. C.R. Mafra, O. Schlotterer, Double-copy structure of one-loop open-string amplitudes. Phys. Rev. Lett. 121(1), 011601 (2018). https://doi.org/10.1103/PhysRevLett.121.011601, [arXiv:1711.09104 [hep-th]]

68. C.R. Mafra, Pure spinor superspace identities for massless four-point kinematic factors. JHEP 04, 093 (2008). https://doi.org/10.1088/1126-6708/2008/04/093, [arXiv:0801.0580 [hep-th]]

69. C.R. Mafra, O. Schlotterer, Two-loop five-point amplitudes of super Yang-Mills and supergrav-ity in pure spinor superspace. JHEP 10, 124 (2015). https://doi.org/10.1007/JHEP10(2015)124, [arXiv:1505.02746 [hep-th]]

70. C.R. Mafra, O. Schlotterer, Cohomology foundations of one-loop amplitudes in pure spinor superspace. [arXiv:1408.3605 [hep-th]]

71. C.R. Mafra, O. Schlotterer, Solution to the nonlinear field equations of ten dimensional supersymmetric Yang-Mills theory. Phys. Rev. D 92(6), 066001 (2015). https://doi.org/10.1103/PhysRevD.92.066001, [arXiv:1501.05562 [hep-th]]

72. O. Schlotterer, S. Stieberger, Motivic multiple zeta values and superstring amplitudes. J. Phys. A 46, 475401 (2013). https://doi.org/10.1088/1751-8113/46/47/475401, [arXiv:1205.1516 [hep-th]]

73. C. Reutenauer, Free Lie Algebras, in London Mathematical Society Monographs. New Series, vol. 7 (The Clarendon Press, Oxford University Press, New York, 1993)

74. E.P. Verlinde, H.L. Verlinde, Chiral bosonization, determinants and the string partition function. Nucl. Phys. B 288, 357 (1987). https://doi.org/10.1016/0550-3213(87)90219-7

75. E.P. Verlinde, H.L. Verlinde, Multiloop calculations in covariant superstring theory. Phys. Lett. B 192, 95-102 (1987). https://doi.org/10.1016/0370-2693(87)91148-8

76. E. D'Hoker, C.R. Mafra, B. Pioline, O. Schlotterer, unpublished

77. M.A. Van Leeuwen, A.M. Cohen, B. Lisser, LiE: A Package for Lie Group Computations (1992)

78. J.P. Harnad, S. Shnider, Constraints and field equations for ten-dimensional super Yang-Mills theory. Commun. Math. Phys. 106, 183 (1986). https://doi.org/10.1007/BF01454971

79. E. Witten, Twistor - like transform in ten-dimensions. Nucl. Phys. B 266, 245-264 (1986). https://doi.org/10.1016/0550-3213(86)90090-8

80. W. Siegel, Superfields in higher dimensional space-time. Phys. Lett. B 80, 220-223 (1979). https://doi.org/10.1016/0370-2693(79)90202-8

81. A. Neveu, J. Scherk, Connection between Yang-Mills fields and dual models. Nucl. Phys. B 36, 155-161 (1972). https://doi.org/10.1016/0550-3213(72)90301-X

82. C.R. Mafra, O. Schlotterer, Tree-level amplitudes from the pure spinor superstring. [arXiv:2210.14241 [hep-th]]

83. R. Lipinski Jusinskas, Nilpotency of the b ghost in the non-minimal pure spinor formalism. JHEP 05, 048 (2013). https://doi.org/10.1007/JHEP05(2013)048, [arXiv:1303.3966 [hep-th]]

84. O. Chandia, The b ghost of the pure spinor formalism is nilpotent. Phys. Lett. B 695, 312-316 (2011). https://doi.org/10.1016/j.physletb.2010.10.058, [arXiv:1008.1778 [hep-th]]

85. J.M. Figueroa-O'Farrill, N = 2 structures in all string theories. J. Math. Phys. 38, 5559-5575 (1997). https://doi.org/10.1063/1.532151, [arXiv:hep-th/9507145 [hep-th]]

86. M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. Commun. Math. Phys. 165, 311-428 (1994). https://doi.org/10.1007/BF02099774, [arXiv:hep-th/9309140 [hep-th]]

87. Y. Geyer, R. Monteiro, R. Stark-Muchão, Superstring loop amplitudes from the field theory limit. Phys. Rev. Lett. 127(21), 211603 (2021). https://doi.org/10.1103/PhysRevLett.127.211603, [arXiv:2106.03968 [hep-th]]

88. B.E.W. Nilsson, Pure spinors as auxiliary fields in the ten-dimensional supersymmetric Yang-Mills theory. Class. Quant. Grav. 3, L41 (1986). https://doi.org/10.1088/0264-9381/3/2/007

89. L.P. Hughston, W.T. Shaw, Classical strings in ten-dimensions. Proc. R. Soc. Lond. A 414, 423 (1987). https://doi.org/10.1098/rspa.1987.0152